M-functions for closed extensions of adjoint pairs of operators with applications to elliptic boundary problems

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Dedicated to the memory of Leonid R. Volevič.

In this paper, we combine results on extensions of operators with recent results on the relation between the M-function and the spectrum, to examine the spectral behaviour of boundary value problems. M-functions are defined for general closed extensions, and associated with realisations of elliptic operators. In particular, we consider both ODE and PDE examples where it is possible for the operator to possess spectral points that can not be detected by the M-function.

1 Introduction

The extension theory for unbounded operators in Hilbert spaces has been studied since at least 1929 when von Neumann discovered the so-called Kreĭn extension. There are many applications of a general extension theory to problems generated by both ODE and PDE examples. In the case of symmetric ODEs the book of Naĭmark [36] characterises the extensions of the minimal operator by means of a Cayley transform between the deficiency spaces and determines all of these extensions by the imposition of explicit boundary conditions. For PDEs, adjoint pairs of second order elliptic operators, their extensions and boundary value problems were studied in the paper of Vishik [43] while Grubb [15] showed that all closed extensions of the minimal operator can be characterised by nonlocal boundary conditions, building on work of Lions and Magenes [29, 30] (cf. also Hörmander [23]) for elliptic operators.

The theory of boundary value spaces (also known as boundary triplets) associated with symmetric operators has its origins in the work of Kočubeĭ [24] and Gorbachuk and Gorbachuk [14] with developments from many authors, (see [6, 25, 27, 28, 35, 37, 39, 41]). In this context, the theory of the Weyl-M-function was developed by Derkach and Malamud [9, 10], where spectral properties of the operator were investigated via the M-function and Kreĭn-type resolvent formulae were established. For adjoint pairs of abstract operators, boundary triplets were introduced by Vainerman [42] and Lyantze and Storozh [31]. Many of the results proved for the symmetric case have subsequently been extended to this situation: see, for instance, Malamud and Mogilevski [32] for adjoint pairs of operators, and Malamud and Mogilevski [33, 34] for adjoint pairs of linear relations. Amrein and Pearson [1] generalised several results from the classical Weyl-m-function for the one-dimensional Sturm-Liouville problem to the case of Schrödinger operators, calling them M-functions, in particular they were able to show nesting results for families of M-functions on spherical exterior domains in $\mathbb{R}^3$. For a recent contribution with applications to PDEs and characterisation of eigenvalues as poles of an operator valued Weyl-M-function,

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we refer the reader to [7]. Further recent developments in this area can be found in [3, 8, 12, 38]. There has also been extensive work on Dirichlet-to-Neumann maps, also sometimes known as Poincaré-Steklov operators, especially in the inverse problems literature. These operators have physical meaning, associating, for instance, a surface current to an applied voltage and are, in some sense, the natural PDE realisation of the abstract M-function which appears in the theory of boundary triplets discussed above.

Systems of PDEs and even ODEs occur naturally in physical applications (reaction-diffusion equations, Maxwell systems, Dirac systems, Lamé systems) and there is much interest in the spectral properties of operators generated by these. In Grubb [17] and Geymonat and Grubb [13] such systems are extensively discussed and *inter alia* points of essential spectrum are characterised by failure of ellipticity of the operator or the boundary condition. An alternative abstract approach for block operator matrices has also been developed (see for example Atkinson et al. [2]).

In this paper, we shall combine results obtained by Grubb on extensions of operators, see for example [15], with recent results obtained by Brown, Marletta, Naboko and Wood [7] on the relation between the M-function and the spectrum, to examine the spectral behaviour of boundary value problems. M-functions are defined for general closed extensions, and associated with realisations of elliptic operators. In particular, we shall consider both ODE and PDE examples where it is possible for the operator to possess spectral points that can not be detected by the M-function (unlike the classical Sturm-Liouville case).

In PDE cases, the kernel of the maximal realisation has infinite dimension; then unbounded operators between boundary spaces must be allowed, and it is important to choose the representations of the boundary mappings in an efficient way. We here rely on the calculus of pseudodifferential operators (ψdo's), as introduced through works of Calderon, Zygmund, Mihlin, Kohn, Nirenberg, Hörmander, Seeley and others around the 1960's, as well as the calculus of pseudodifferential boundary operators (ψdbo's) introduced by Boutet de Monvel [4, 5] and applied and extended by Grubb [17]– [21] and others.

Plan of the paper. Section 2 contains a discussion of the abstract theory. We begin by recalling the universal parametrization of closed extensions $\widetilde{A}$ established in [15], based on an invertible reference operator A_β, and show how it applies to operators $\widetilde{A}-\lambda$ (by use of techniques from [17]), giving rise to a Kreĭn resolvent formula and characterisations of kernels and ranges, in terms of an abstract boundary operator $T^\lambda : V_\lambda \to W_{\bar\lambda}$. Next, we connect this with the boundary triplets theory, as presented in [7]. We first show that for the realisation A_B defined by a boundary condition $\Gamma_1 u = B\Gamma_0 u$ (for a special choice of Γ_0, Γ_1), the holomorphic operator family $M_B(\lambda)$ defined for $\lambda \in \varrho(A_B)$ is homeomorphic to the inverse of the holomorphic family T^λ defined for $\lambda \in \varrho(A_\beta)$, when both exist. This takes care of a special class of boundary conditions. The idea is now developed further to include general extensions by considering mappings between subspaces, as in [15]. In the present context, this replaces the need to work with relations. M-functions are now defined for all closed extensions $\widetilde{A}$, and the operator families $M_{\widetilde{A}}(\lambda)$ and T^λ together describe spectral properties of the operator.

In Section 3, the ideas are implemented for realisations of elliptic operators on smooth domains Ω in n-space. For second-order strongly elliptic operators it is shown in detail how T^λ and $M_{\widetilde{A}}(\lambda)$, for Neumann-type boundary conditions $\gamma_1 u = C\gamma_0 u$, are carried over to mappings L^λ and $M_L(\lambda)$ between Sobolev spaces over $\partial\Omega$. The general closed realisations give rise to operator families L_1^λ and $M_{L_1}(\lambda)$ between closed subspaces of $L_2(\partial\Omega)$. For systems and higher-order operators, the normal elliptic boundary conditions give rise to M- and L-functions between products of Sobolev spaces over $\partial\Omega$.

Section 4 addresses the inverse question: Are the spectral properties fully described by $M_{\widetilde{A}}(\lambda)$ and T^λ? The answer is in the affirmative for $\lambda \in \varrho(\widetilde{A}) \cup \varrho(A_\beta)$, and this is sufficiently informative in many situations. But it is not so in general: We show, both by a PDE and an ODE matrix example, that there exist cases where the M-function is holomorphic across points in the essential spectrum of $\widetilde{A}$ (and of A_β).

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2 Universal parametrization and M-functions

2.1 A universal parametrization of closed extensions

It is assumed in this paper that there is given a pair of closed, densely defined operators $A_{\min}$ and $A'_{\min}$ in a Hilbert space H (a so-called adjoint pair) such that the adjoint of $A'_{\min}$ is an extension of $A_{\min}$ and the adjoint of

$A_{\min}$ is an extension of $A'_{\min}$; we call these adjoints $A_{\max}$ resp. $A'_{\max}$. Moreover we assume that there is given a closed, densely defined operator A_β lying between $A_{\min}$ and $A_{\max}$ and having a bounded everywhere defined inverse. Thus we get:

$$\begin{aligned} &A_{\min} \subset A_\beta \subset A_{\max}, \quad 0 \in \varrho(A_\beta), \quad A_{\max} = (A'_{\min})^*, \\ &A'_{\min} \subset A^*_\beta \subset A'_{\max}, \quad 0 \in \varrho(A^*_\beta), \quad A'_{\max} = A^*_{\min}; \end{aligned} \tag{2.1}$$

here $\varrho(B)$ denotes the resolvent set of B. We call A_β the reference operator. Let $\mathcal{M}$ and $\mathcal{M}'$ denote the sets of operators $\widetilde{A}$ lying between $A_{\min}$ and $A_{\max}$, resp. $\widetilde{A}'$ lying between $A'_{\min}$ and $A'_{\max}$. We write Au for $\widetilde{A}u$ when $\widetilde{A} \in \mathcal{M}$, resp. $A'v$ for $\widetilde{A}'v$ when $\widetilde{A}' \in \mathcal{M}'$. When U is a closed subspace of H, we denote by f_U the orthogonal projection of f onto U; the projection map is denoted pr_U.

Denote also

$$\ker A_{\max} = Z, \quad \ker A'_{\max} = Z'; \tag{2.2}$$

and let

$$\begin{aligned} \operatorname{pr}_\beta = A_\beta^{-1} A_{\max} : D(A_{\max}) \to D(A_\beta), \quad \operatorname{pr}_\zeta = I - \operatorname{pr}_\beta : D(A_{\max}) \to Z, \\ \operatorname{pr}_{\beta'} = (A^*_\beta)^{-1} A'_{\max} : D(A'_{\max}) \to D(A^*_\beta), \quad \operatorname{pr}_{\zeta'} = I - \operatorname{pr}_{\beta'} : D(A'_{\max}) \to Z'. \end{aligned} \tag{2.3}$$

Then pr_β and pr_ζ, resp. $\operatorname{pr}_{\beta'}$ and $\operatorname{pr}_{\zeta'}$, are complementary projections defining the direct sum decompositions

$$D(A_{\max}) = D(A_\beta) \dot{+} Z, \text{ resp. } D(A'_{\max}) = D(A^*_\beta) \dot{+} Z'. \tag{2.4}$$

We also write $\operatorname{pr}_\beta u = u_\beta$, $\operatorname{pr}_\zeta u = u_\zeta$, etc.

The above statements are verified in [15], which also showed the abstract Green's formula

$$(Au, v) - (u, A'v) = ((Au)_{Z'}, v_{\zeta'}) - (u_\zeta, (A'v)_Z), \text{ for } u \in D(A_{\max}), v \in D(A'_{\max}); \tag{2.5}$$

and we recall that in that paper, all the closed operators in $\mathcal{M}$ were characterised by abstract boundary conditions:

Theorem 2.1 *There is a one-to-one correspondence between all closed operators $\widetilde{A} \in \mathcal{M}$ and all operators $T : V \to W$, where V and W are closed subspaces of Z resp. Z', and T is closed with domain $D(T)$ dense in V. Here $T : V \to W$ is defined from $\widetilde{A}$ by*

$$\begin{gathered} D(T) = \operatorname{pr}_\zeta D(\widetilde{A}), \quad V = \overline{D(T)}, \quad W = \overline{\operatorname{pr}_{\zeta'} D(\widetilde{A}^*)}, \\ Tu_\zeta = (Au)_W \textit{ for } u \in D(\widetilde{A}); \end{gathered} \tag{2.6}$$

and $\widetilde{A}$ is defined from $T : V \to W$ by

$$D(\widetilde{A}) = \{u \in D(A_{\max}) \mid u_\zeta \in D(T), \, (Au)_W = Tu_\zeta\}; \tag{2.7}$$

it can also be described by

$$u \in D(\widetilde{A}) \iff u = v + z + A_\beta^{-1}(Tz + f), v \in D(A_{\min}), z \in D(T), f \in Z' \ominus W. \tag{2.8}$$

All closed subspaces $V \subset Z$ and $W \subset Z'$ and all closed densely defined operators $T : V \to W$ are reached in this correspondence.

When $\widetilde{A}$ corresponds to $T : V \to W$,

$$\begin{aligned} &\ker \widetilde{A} = \ker T, \\ &\operatorname{ran} \widetilde{A} = \operatorname{ran} T + (H \ominus W), \end{aligned} \tag{2.9}$$

orthogonal sum. In particular, $\widetilde{A}$ is Fredholm if and only if T is so, with the same kernel and cokernel.

If $\widetilde{A}$, hence also T, is injective, the inverse satisfies

$$\widetilde{A}^{-1} = A_\beta^{-1} + T^{-1} \operatorname{pr}_W, \textit{ defined on } \operatorname{ran} \widetilde{A}. \tag{2.10}$$

The adjoint $\widetilde{A}^*$ corresponds to $T^* : W \to V$ in the analogous way. In particular, in the case where $A_{\min} = A'_{\min}$ and A_β is selfadjoint (called the symmetric case), $\widetilde{A}$ is selfadjoint if and only if $V = W$ and T is selfadjoint.

Remark 2.2 The characterisation is related to that of Vishik [43], but differs in an important way: Vishik was concerned with *normally solvable* operators $\widetilde{A}$ (those with closed range), and his operators between subspaces of Z and Z' map in the opposite direction of those in [15], covering only a subset of them. In contrast, the theory in [15] allowed the characterisation of *all* closed operators in $\mathcal{M}$.

There are also some results in [15, Section II.3] on non-closed extensions.

Now consider the situation where a spectral parameter $\lambda \in \mathbb{C}$ is subtracted from the operators in $\mathcal{M}$. When $\lambda \in \varrho(A_\beta)$, we have a similar situation as above:

$$A_{\min} - \lambda \subset A_\beta - \lambda \subset A_{\max} - \lambda, \quad A'_{\min} - \bar\lambda \subset A^*_\beta - \bar\lambda \subset A'_{\max} - \bar\lambda, \tag{2.11}$$

and we use the notation $\mathcal{M}_\lambda$, $\mathcal{M}'_{\bar\lambda}$, and

$$\begin{aligned} \ker(A_{\max} - \lambda) &= Z_\lambda, \quad \ker(A'_{\max} - \bar\lambda) = Z'_{\bar\lambda};\\ \operatorname{pr}^\lambda_\beta = (A_\beta - \lambda)^{-1}(A - \lambda), &\quad \operatorname{pr}^{\bar\lambda}_{\beta'} = (A^*_\beta - \bar\lambda)^{-1}(A' - \bar\lambda),\\ \operatorname{pr}^\lambda_\zeta = I - \operatorname{pr}^\lambda_\beta, &\quad \operatorname{pr}^{\bar\lambda}_{\zeta'} = I - \operatorname{pr}^{\bar\lambda}_{\beta'}\,. \end{aligned} \tag{2.12}$$

Then we have an immediate corollary of Theorem 2.1:

Corollary 2.3 *Let $\lambda \in \varrho(A_\beta)$. There is a* 1–1 *correspondence between the closed operators $\widetilde{A} - \lambda$ in $\mathcal{M}_\lambda$ and the closed, densely defined operators $T^\lambda : V_\lambda \to W_{\bar\lambda}$, where V_λ and $W_{\bar\lambda}$ are closed subspaces of Z_λ resp. $Z'_{\bar\lambda}$; here*

$$\begin{aligned} &D(T^\lambda) = \operatorname{pr}^\lambda_\zeta D(\widetilde{A}), \quad V_\lambda = \overline{D(T^\lambda)}, \quad W_{\bar\lambda} = \overline{\operatorname{pr}^{\bar\lambda}_{\zeta'} D(\widetilde{A}^*)},\\ &T^\lambda u^\lambda_\zeta = ((A - \lambda)u)_{W_{\bar\lambda}} \text{ for } u \in D(\widetilde{A}),\\ &D(\widetilde{A}) = \{u \in D(A_{\max}) \mid u^\lambda_\zeta \in D(T^\lambda),\ ((A - \lambda)u)_{W_{\bar\lambda}} = T^\lambda u^\lambda_\zeta\}. \end{aligned} \tag{2.13}$$

In this correspondence,

$$\begin{aligned} &\ker(\widetilde{A} - \lambda) = \ker T^\lambda,\\ &\operatorname{ran}(\widetilde{A} - \lambda) = \operatorname{ran} T^\lambda + (H \ominus W_{\bar\lambda}), \end{aligned} \tag{2.14}$$

orthogonal sum. In particular, if $\lambda \in \varrho(\widetilde{A})$,

$$(\widetilde{A} - \lambda)^{-1} = (A_\beta - \lambda)^{-1} + \mathrm{i}_{V_\lambda \to H}(T^\lambda)^{-1} \operatorname{pr}_{W_{\bar\lambda}}\,. \tag{2.15}$$

Here $\mathrm{i}_{X\to Y}$ denotes the injection $X \hookrightarrow Y$.

The formula (2.15) can be regarded as a universal “Kreĭn resolvent formula”. It relates the resolvents of the arbitrary operator $\widetilde{A}$ and the reference operator A_β in a straightforward way, such that information on the spectrum of $\widetilde{A}$ can be deduced from information on T^λ. Note also the formulas in (2.14), which give not only a correspondence between kernel dimensions and range codimensions, but an identification between kernels and cokernels themselves.

The resolvent of $\widetilde{A}$ was studied in [17], from which we extract the following additional information. Define, for $\lambda \in \varrho(A_\beta)$, the bounded operators on H:

$$\begin{aligned} &E^\lambda = A_{\max}(A_\beta - \lambda)^{-1} = I + \lambda(A_\beta - \lambda)^{-1},\\ &F^\lambda = (A_{\max} - \lambda)A_\beta^{-1} = I - \lambda A_\beta^{-1},\\ &E'^{\bar\lambda} = A'_{\max}(A^*_\beta - \bar\lambda)^{-1} = I + \bar\lambda(A^*_\beta - \bar\lambda)^{-1} = (E^\lambda)^*,\\ &F'^{\bar\lambda} = (A'_{\max} - \bar\lambda)(A^*_\beta)^{-1} = I - \bar\lambda(A^*_\beta)^{-1} = (F^\lambda)^*; \end{aligned} \tag{2.16}$$

then E^λ and F^λ are inverses of one another, and so are $E'^{\bar\lambda}$ and $F'^{\bar\lambda}$. In particular, the operators restrict to homeomorphisms

$$\begin{aligned} E^\lambda_Z &: Z \xrightarrow{\sim} Z_\lambda, \quad F^\lambda_Z : Z_\lambda \xrightarrow{\sim} Z, \\ E'^{\bar\lambda}_{Z'} &: Z' \xrightarrow{\sim} Z'_{\bar\lambda}, \quad F'^{\bar\lambda}_{Z'} : Z'_{\bar\lambda} \xrightarrow{\sim} Z'. \end{aligned} \tag{2.17}$$

Moreover, for $u \in D(A_{\max})$, $v \in D(A'_{\max})$,

$$\begin{aligned} \operatorname{pr}^\lambda_\zeta u &= E^\lambda \operatorname{pr}_\zeta u, \quad \operatorname{pr}^\lambda_\beta u = \operatorname{pr}_\beta u - \lambda(A_\beta - \lambda)^{-1} \operatorname{pr}_\zeta u, \\ \operatorname{pr}^{\bar\lambda}_{\zeta'} v &= E'^{\bar\lambda} \operatorname{pr}_{\zeta'} v, \quad \operatorname{pr}^{\bar\lambda}_{\beta'} v = \operatorname{pr}_{\beta'} v - \bar\lambda(A^*_\beta - \bar\lambda)^{-1} \operatorname{pr}_{\zeta'} v. \end{aligned} \tag{2.18}$$

This was shown in [17, Sect. 2], in the symmetric case, and the (elementary) proofs extend verbatim to the general case. Similar mappings occur frequently in the literature on extensions. The following theorem extends [17, Prop. 2.6], to the non-symmetric situation, with practically the same proof:

Theorem 2.4 *For $\lambda \in \varrho(A_\beta)$, define the operator G^λ from Z to Z' by*

$$G^\lambda z = -\lambda \operatorname{pr}_{Z'} E^\lambda z, \quad z \in Z. \tag{2.19}$$

Then

$$\begin{gathered} D(T^\lambda) = E^\lambda D(T), \quad V_\lambda = E^\lambda V, \quad W_{\bar\lambda} = E'^{\bar\lambda} W, \\ (T^\lambda E^\lambda v, E'^{\bar\lambda} w) = (Tv, w) + (G^\lambda v, w), \text{ for } v \in D(T),\ w \in W. \end{gathered} \tag{2.20}$$

P r o o f. The first line in (2.20) follows from (2.13) in view of (2.18). The second line in calculated as follows: For $u \in D(\widetilde A)$, $w \in W$,

$$\begin{aligned} (Tu_\zeta, w) &= (Au, w) = (Au, F'^{\bar\lambda} E'^{\bar\lambda} w) = (F^\lambda Au, E'^{\bar\lambda} w) \\ &= ((A-\lambda)(A_\beta)^{-1} Au, E'^{\bar\lambda} w) = ((A-\lambda)u_\beta, E'^{\bar\lambda} w) \\ &= ((A-\lambda)u, E'^{\bar\lambda} w) - ((A-\lambda)u_\zeta, E'^{\bar\lambda} w) \\ &= (T^\lambda u^\lambda_\zeta, E'^{\bar\lambda} w) + (\lambda u_\zeta, E'^{\bar\lambda} w) = (T^\lambda E^\lambda u_\zeta, E'^{\bar\lambda} w) + (\lambda E^\lambda u_\zeta, w). \end{aligned}$$

This shows the equation in (2.20) when we set $u_\zeta = v$. □

Denote by E^λ_V the restriction of E^λ to a mapping from V to V_λ, with inverse F^λ_V, and let similarly $E'^{\bar\lambda}_W$ be the restriction of $E'^{\bar\lambda}$ to a mapping from W to $W_{\bar\lambda}$, with inverse $F'^{\bar\lambda}_W$. Then the second line of (2.20) can be written

$$(E'^{\bar\lambda}_W)^* T^\lambda E^\lambda_V = T + G^\lambda_{V,W} \text{ on } D(T) \subset V, \tag{2.21}$$

where

$$G^\lambda_{V,W} = \operatorname{pr}_W G^\lambda \operatorname{i}_{V\to Z}. \tag{2.22}$$

Equivalently,

$$T^\lambda = (F'^{\bar\lambda}_W)^* (T + G^\lambda_{V,W}) F^\lambda_V \text{ on } D(T^\lambda) \subset V_\lambda. \tag{2.23}$$

Then the Kreĭn resolvent formula (2.15) can be made more explicit as follows:

Corollary 2.5 *When $\lambda \in \varrho(\widetilde A) \cap \varrho(A_\beta)$, T^λ is invertible, and*

$$(T^\lambda)^{-1} = E^\lambda_V (T + G^\lambda_{V,W})^{-1} (E'^{\bar\lambda}_W)^*. \tag{2.24}$$

Hence

$$(\widetilde A - \lambda)^{-1} = (A_\beta - \lambda)^{-1} + \operatorname{i}_{V_\lambda \to H} E^\lambda_V (T + G^\lambda_{V,W})^{-1} (E'^{\bar\lambda}_W)^* \operatorname{pr}_{W_{\bar\lambda}}. \tag{2.25}$$

P r o o f. (2.24) follows from (2.23) by inversion, and insertion in (2.15) shows (2.25). □

Note that $G^\lambda_{V,W}$ depends in a simple way on V and W and is independent of T.

2.2 Connections between the universal parametrization and boundary triplets

The setting for boundary triplets in the non-symmetric case is the following, according to [7] (with reference to [31], [34]): $A_{\min}$, $A_{\max}$, $A'_{\min}$ and $A'_{\max}$ are given as in the beginning of Section 2.1, and there is given a pair of Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and two pairs of "boundary operators"

$$\begin{pmatrix}\Gamma_1\\ \\ \Gamma_0\end{pmatrix} : D(A_{\max}) \to \begin{matrix}\mathcal{H}\\ \times,\\ \mathcal{K}\end{matrix} \quad \begin{pmatrix}\Gamma'_1\\ \\ \Gamma'_0\end{pmatrix} : D(A'_{\max}) \to \begin{matrix}\mathcal{K}\\ \times,\\ \mathcal{H}\end{matrix} \tag{2.26}$$

bounded with respect to the graph norm and surjective, such that

$$(Au, v) - (u, A'v) = (\Gamma_1 u, \Gamma'_0 v)_{\mathcal{H}} - (\Gamma_0 u, \Gamma'_1 v)_{\mathcal{K}},\ \text{all } u \in D(A_{\max}), v \in D(A'_{\max}), \tag{2.27}$$

and

$$D(A_{\min}) = D(A_{\max}) \cap \ker \Gamma_1 \cap \ker \Gamma_0, \quad D(A'_{\min}) = D(A'_{\max}) \cap \ker \Gamma'_1 \cap \ker \Gamma'_0.$$

Note that under the assumption of (2.1), the choice

$$\mathcal{H} = Z',\ \mathcal{K} = Z,\ \Gamma_1 = \operatorname{pr}_{Z'} A_{\max},\ \Gamma_0 = \operatorname{pr}_{\zeta},\ \Gamma'_1 = \operatorname{pr}_Z A'_{\max},\ \Gamma'_0 = \operatorname{pr}_{\zeta'}, \tag{2.28}$$

defines in view of (2.4) and (2.5) a boundary triplet.

Following [7], the boundary triplet is used to define operators $A_B \in \mathcal{M}$ and $A'_{B'} \in \mathcal{M}'$ for any pair of operators $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $B' \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ by

$$D(A_B) = \ker(\Gamma_1 - B\Gamma_0), \quad D(A'_{B'}) = \ker(\Gamma'_1 - B'\Gamma'_0). \tag{2.29}$$

In order to discuss resolvents, [7] assumes that $\varrho(A_B) \neq \emptyset$, which means that the situation with existence of an invertible A_β as in (2.1) can be obtained at least after subtraction of a spectral parameter λ_0. Thus we can build on the full assumption (2.1) from now on. (Theorem 2.1 shows that there is an abundance of different invertible operators in $\mathcal{M}$ then.)

Definition 2.6 For $\lambda \in \varrho(A_B)$, the M-function $M_B(\lambda)$ is defined by

$$M_B(\lambda) : \operatorname{ran}(\Gamma_1 - B\Gamma_0) \to \mathcal{K}, \quad M_B(\Gamma_1 - B\Gamma_0)u = \Gamma_0 u \text{ for all } u \in Z_\lambda;$$

and for $\lambda \in \varrho(A'_{B'})$, the M-function $M'_{B'}(\lambda)$ is defined similarly by

$$M'_{B'}(\lambda) : \operatorname{ran}(\Gamma'_1 - B'\Gamma'_0) \to \mathcal{H}, \quad M'_{B'}(\Gamma'_1 - B'\Gamma'_0)v = \Gamma'_0 v \text{ for all } v \in Z'_\lambda.$$

It is shown in [7] that when $\varrho(A_B) \neq \emptyset$,

$$(A_B)^* = A'_{B^*}. \tag{2.30}$$

We shall set all this in relation to the universal parametrization, when the boundary triplet is chosen as in (2.28). *We assume* (2.28) *from now on.*

Concerning A_B, note that B is taken as a bounded operator from Z to Z', and that

$$D(A_B) = \{u \in D(A_{\max}) \mid (Au)_{Z'} = Bu_\zeta\},$$

by definition. This shows that for the operator $T : V \to W$ that A_B corresponds to by Theorem 2.1,

$$V = Z, \quad W = Z', \quad T = B.$$

Note that (2.30) follows from Theorem 2.1.

When Z and Z' are finite dimensional, all operators B will be bounded. But in the case where $\dim Z = \dim Z' = \infty$, Theorem 2.1 shows that unbounded T's must be allowed, to cover general extensions. Therefore we in the following take B closed, densely defined and possibly unbounded, and define A_B by

$$D(A_B) = \{u \in D(A_{\max}) \mid u_\zeta \in D(B), (Au)_{Z'} = Bu_\zeta\}. \tag{2.31}$$

Lemma 2.7 $\operatorname{ran}(\Gamma_1 - B\Gamma_0) = Z'$. *In fact, any* $f \in Z'$ *can be written as* $f = (\Gamma_1 - B\Gamma_0)v = \Gamma_1 v$ *for* $v \in D(A_\beta)$ *taken equal to* $A_\beta^{-1} f$.

P r o o f. Let v run through $D(A_\beta)$. Then $\Gamma_0 v = \operatorname{pr}_\zeta v = 0 \in D(B)$, and Av runs through $H = (\operatorname{ran} A_{\min}) \oplus Z'$, so $\Gamma_1 v - B\Gamma_0 v = (Av)_{Z'}$ runs through Z'. In other words, we can take $v = A_\beta^{-1} f$, for any $f \in Z'$. □

Lemma 2.8 *For any* $\lambda \in \varrho(A_B)$, $M_B(\lambda)$ *is well-defined as a mapping from* Z' *to* Z *by*

$$M_B(\lambda)(\Gamma_1 - B\Gamma_0)u = \Gamma_0 u \text{ for all } u \in Z_\lambda \text{ with } \Gamma_0 u \in D(B),$$

also when $D(B)$ *is a subset of* Z. *In fact,*

$$M_B(\lambda) = \operatorname{pr}_\zeta (I - (A_B - \lambda)^{-1}(A_{\max} - \lambda)) A_\beta^{-1} \, \mathrm{i}_{Z' \to H} \,. \tag{2.32}$$

P r o o f. In the defining equation, we can now only allow those $u = z^\lambda \in Z_\lambda$ for which $\operatorname{pr}_\zeta z^\lambda \in D(B)$.

We first show: When $f = (\Gamma_1 - B\Gamma_0)v$ for $v = A_\beta^{-1} f$ as in Lemma 2.7, then there is a $z^\lambda \in Z_\lambda$ with $\Gamma_0 z^\lambda \in D(B)$ such that

$$f = (\Gamma_1 - B\Gamma_0) z^\lambda. \tag{2.33}$$

For, let $x = (A_B - \lambda)^{-1}(A - \lambda)v$, then $v - x \in Z_\lambda$. Moreover, since $x \in D(A_B)$, $\operatorname{pr}_\zeta x \in D(B)$ in view of (2.31), and $\operatorname{pr}_\zeta v = 0 \in D(B)$, as already noted. Hence

$$(\Gamma_1 - B\Gamma_0)(v - x) = (\Gamma_1 - B\Gamma_0)v = f;$$

so we can take $z^\lambda = v - x$. We conclude:

$$\{(\Gamma_1 - B\Gamma_0) z^\lambda \mid z^\lambda \in Z_\lambda,\ \operatorname{pr}_\zeta z^\lambda \in D(B)\} = Z'.$$

Next, we note that the z^λ in (2.33) is uniquely determined from f. For, if $f = 0$ and z^λ solves (2.33), then $z^\lambda \in D(A_B) = D(A_B - \lambda)$, which is linearly independent from Z_λ since $\lambda \in \varrho(A_B)$, so $z^\lambda = 0$.

Thus we can for any $f \in Z'$ set $M_B(\lambda) f = \operatorname{pr}_\zeta z^\lambda$ where z^λ is the unique solution of (2.33); this defines a linear mapping $M_B(\lambda)$ from Z' to Z. The procedures used above to construct $M_B(\lambda)$ are summed up in (2.32). □

Since B is closed, $M_B(\lambda)$ is closed, hence continuous, as a mapping from Z' to Z.

For a further analysis of $M_B(\lambda)$, assume $\lambda \in \varrho(A_\beta)$. Then the maps E^λ, F^λ etc. in (2.16) are defined. Let $z^\lambda \in Z_\lambda$, and consider the defining equation

$$M_B(\lambda)((Az^\lambda)_{Z'} - B \operatorname{pr}_\zeta z^\lambda) = \operatorname{pr}_\zeta z^\lambda; \tag{2.34}$$

where $\operatorname{pr}_\zeta z^\lambda$ is required to lie in $D(B)$. By (2.17) and (2.18), there is a unique $z \in Z$ such that $z^\lambda = E_Z^\lambda z$; in fact

$$z^\lambda = E_Z^\lambda z = \operatorname{pr}_\zeta^\lambda z; \quad z = F_Z^\lambda z^\lambda = \operatorname{pr}_\zeta z^\lambda,$$

so the requirement is that $z^\lambda \in E_Z^\lambda D(B)$.

Writing (2.34) in terms of z, and using that $Az^\lambda = \lambda z^\lambda$, we find:

$$\begin{aligned} z = \operatorname{pr}_\zeta z^\lambda &= M_B(\lambda)((Az^\lambda)_{Z'} - Bz) = M_B(\lambda)((\lambda z^\lambda)_{Z'} - Bz) \\ &= M_B(\lambda)((\lambda E^\lambda z)_{Z'} - Bz) = M_B(\lambda)(-G^\lambda - B)z, \end{aligned} \tag{2.35}$$

cf. (2.19). Here we observe that the operator to the right of $M_B(\lambda)$ equals T^λ from Section 2.1 up to homeomorphisms:

$$-G^\lambda - B = -(G^\lambda + T) = -(E'^{\bar\lambda}_{Z'})^* T^\lambda E_Z^\lambda, \tag{2.36}$$

by (2.21); here T^λ is invertible from $E_Z^\lambda D(B)$ onto Z'_λ. We conclude that $M_B(\lambda)$ is the inverse of the operator in (2.36). We have shown:

Theorem 2.9 *When the boundary triplet is chosen as in* (2.28) *and* $\lambda \in \varrho(A_B) \cap \varrho(A_\beta)$, $-M_B(\lambda)$ *equals the inverse of* $B + G^\lambda = T + G^\lambda$, *also equal to the inverse of* T^λ *modulo homeomorphisms:*

$$-M_B(\lambda)^{-1} = B + G^\lambda = T + G^\lambda = (E'^{\bar\lambda}_{Z'})^* T^\lambda E^\lambda_Z. \tag{2.37}$$

In particular, $M_B(\lambda)$ *has range* $D(B)$.

With this insight we have access to the straightforward resolvent formula (2.25), which implies in this case:

Corollary 2.10 *For* $\lambda \in \varrho(A_B) \cap \varrho(A_\beta)$,

$$(A_B - \lambda)^{-1} = (A_\beta - \lambda)^{-1} - \mathrm{i}_{Z_\lambda \to H}\, E^\lambda_Z M_B(\lambda) (E'^{\bar\lambda}_{Z'})^* \operatorname{pr}_{Z'_{\bar\lambda}}. \tag{2.38}$$

Remark 2.11 Other Kreĭn-type resolvent formulae for realisations in the general framework of relations can be found in [34, Section 5.2].

We also have the direct link between null-spaces and ranges (2.14), when merely $\lambda \in \varrho(A_\beta)$.

Corollary 2.12 *For any* $\lambda \in \varrho(A_\beta)$,

$$\begin{aligned} \ker(A_B - \lambda) &= E^\lambda_Z \ker(B + G^\lambda), \\ \operatorname{ran}(A_B - \lambda) &= (F'^{\bar\lambda}_{Z'})^* \operatorname{ran}(B + G^\lambda) + \operatorname{ran}(A_{\min} - \lambda). \end{aligned} \tag{2.39}$$

For $\lambda \in \varrho(A_\beta)$, this adds valuable information to the results from [7] on the connection between eigenvalues of A_B and poles of $M_B(\lambda)$.

The analysis moreover implies that $M_B(\lambda)$ and $M'_{B^*}(\bar\lambda)$ are adjoints, at least when $\lambda \in \varrho(A_\beta)$.

Observe that $T + G^\lambda$ (and T^λ) is well-defined for all $\lambda \in \varrho(A_\beta)$, whereas $M_B(\lambda)$ is well-defined for all $\lambda \in \varrho(A_B)$; the latter fact is useful for other purposes. In this way, the two operator families complement each other, and, together, contain much spectral information.

It is noteworthy that the widely studied boundary triplets theory leads to an operator family whose elements are inverses of elements of the operator family generated by Theorem 2.1 – compare with Remark 2.2 on the connection with Vishik's theory.

2.3 The M-function for arbitrary closed extensions

The above considerations do not fully use the potential of Corollary 2.3, Theorem 2.4 and Corollary 2.5, which allow much more general boundary operators $T : V \to W$. But, inspired by the result in Theorem 2.9, we can in fact establish useful M-functions in all these other cases, namely homeomorphic to the inverses of the operators T^λ that exist for $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\beta)$, and extended to exist for all $\lambda \in \varrho(\widetilde{A})$.

Theorem 2.13 *Let* $\widetilde{A}$ *be an arbitrary closed densely defined operator between* $A_{\min}$ *and* $A_{\max}$, *and let* $T : V \to W$ *be the corresponding operator according to Theorem* 2.1. *For any* $\lambda \in \varrho(\widetilde{A})$ *there is a bounded operator* $M_{\widetilde{A}}(\lambda) : W \to V$, *depending holomorphically on* $\lambda \in \varrho(\widetilde{A})$, *such that when* $\lambda \in \varrho(A_\beta)$, $-M_{\widetilde{A}}(\lambda)$ *is the inverse of* $T + G^\lambda_{V,W}$, *and is homeomorphic to* T^λ *(as defined in Section* 2.1). *It satisfies*

$$M_{\widetilde{A}}(\lambda)((Az^\lambda)_W - T \operatorname{pr}_\zeta z^\lambda) = \operatorname{pr}_\zeta z^\lambda, \tag{2.40}$$

for all $z^\lambda \in Z_\lambda$ *such that* $\operatorname{pr}_\zeta z^\lambda \in D(T)$. *Its definition extends to all* $\lambda \in \varrho(\widetilde{A})$ *by the formula*

$$M_{\widetilde{A}}(\lambda) = \operatorname{pr}_\zeta \big(I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda)\big) A_\beta^{-1}\, \mathrm{i}_{W \to H} \tag{2.41}$$

In particular, the resolvent formula

$$(\widetilde{A} - \lambda)^{-1} = (A_\beta - \lambda)^{-1} - \mathrm{i}_{V_\lambda \to H}\, E^\lambda_V M_{\widetilde{A}}(\lambda)(E'^{\bar\lambda}_W)^* \operatorname{pr}_{W_{\bar\lambda}} \tag{2.42}$$

holds when $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\beta)$. *For all* $\lambda \in \varrho(A_\beta)$,

$$\begin{aligned} \ker(\widetilde{A} - \lambda) &= E^\lambda_V \ker(T + G^\lambda_{V,W}), \\ \operatorname{ran}(\widetilde{A} - \lambda) &= (F'^{\bar\lambda}_W)^* \operatorname{ran}(T + G^\lambda_{V,W}) + H \ominus W_{\bar\lambda}. \end{aligned} \tag{2.43}$$

P r o o f. Following the lines of proofs of Lemma 2.7 and 2.8, we define $M_{\widetilde{A}}(\lambda)$ satisfying (2.40) as follows: Let $f \in W$. Let $v = A_\beta^{-1} f$; then $\mathrm{pr}_\zeta\, v = 0 \in D(T)$, and

$$(Av)_W - T\,\mathrm{pr}_\zeta\, v = Av = f. \tag{2.44}$$

Next, let $x = (\widetilde{A} - \lambda)^{-1}(A - \lambda)v$, then $z^\lambda = v - x$ lies in Z_λ and satisfies $\mathrm{pr}_\zeta\, z^\lambda \in D(T)$ (since $\mathrm{pr}_\zeta\, v = 0$ and $\mathrm{pr}_\zeta\, x \in D(T)$, cf. (2.6)). This z^λ satisfies

$$(Az^\lambda)_W - T\,\mathrm{pr}_\zeta\, z^\lambda = f, \tag{2.45}$$

in view of (2.44) and the fact that $x \in D(\widetilde{A})$.

Next, observe that for any vector $z^\lambda \in Z_\lambda$ with $\mathrm{pr}_\zeta\, z^\lambda \in D(T)$ such that (2.45) holds, $f = 0$ implies $z^\lambda = 0$, since such a z^λ lies in the two linearly independent spaces $D(\widetilde{A} - \lambda)$ and Z_λ. So there is indeed a mapping from f to $\mathrm{pr}_\zeta\, z^\lambda$ solving (2.45), for any $f \in W$, defining $M_{\widetilde{A}}(\lambda)$. It is described by (2.41). The holomorphicity in $\lambda \in \varrho(\widetilde{A})$ is seen from this formula.

The mapping is connected with T^λ (cf. Corollary 2.3) as follows:

When $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\beta)$, then $z = \mathrm{pr}_\zeta\, z^\lambda = F_V^\lambda z^\lambda$, and $z^\lambda = E_V^\lambda z$, so the vectors z^λ with $\mathrm{pr}_\zeta\, z^\lambda \in D(T)$ constitute the space $E_V^\lambda D(T)$. Calculating as in (2.35) we then find that

$$\begin{aligned} z = \mathrm{pr}_\zeta\, z^\lambda &= M_{\widetilde{A}}(\lambda)((Az^\lambda)_W - Tz) = M_{\widetilde{A}}(\lambda)((\lambda z^\lambda)_W - Tz) \\ &= M_{\widetilde{A}}(\lambda)((\lambda E^\lambda z)_W - Bz) = M_{\widetilde{A}}(\lambda)(-G_{V,W}^\lambda - T)z, \end{aligned}$$

so $M_{\widetilde{A}}(\lambda)$ is the inverse of $-(T + G_{V,W}^\lambda) : D(T) \to W$. The remaining statements follow from Corollary 2.3 and Corollary 2.5. □

Note that when $M_{\widetilde{A}}(\lambda)$ is considered in a neighbourhood of a spectral point of $\widetilde{A}$ in $\varrho(A_\beta)$, then we have not only information on the possibility of a pole of $M_{\widetilde{A}}(\lambda)$, but an inverse T^λ, from which $\ker(\widetilde{A} - \lambda)$ and $\mathrm{ran}(\widetilde{A} - \lambda)$ can be read off.

3 Applications to elliptic partial differential operators

3.1 Preliminaries

For elliptic operators A defined over an open subset Ω of $\mathbb{R}^n$, $n > 1$, the null-space of the maximal realisation is infinite dimensional, so that there is much more freedom of choice of boundary spaces and mappings than in ODE cases. It is necessary to allow unbounded operators between boundary spaces to obtain a theory covering the well-known cases. Moreover, there is the problem of regularity of domains: For a given realisation $\widetilde{A}$ representing a boundary condition, it is not always certain that $\widetilde{A}^*$ represents an analogous boundary condition, but this can often be assured if $D(\widetilde{A})$ is known to be contained in the most regular Sobolev space $H^m(\Omega)$, where m is the order of A; this holds when the boundary condition is *elliptic*.

The theory of pseudodifferential boundary operators (Boutet de Monvel [4], [5], and e.g. Grubb [20]– [22]) is known as an efficient tool in the treatment of boundary value problems on smooth sets (we call it the ψdbo calculus for short, similarly to the customary use of ψdo for pseudodifferential operator). A guiding principle in the construction of general theories would therefore be to make it possible to use the ψdbo calculus in applications to concrete operators. The ψdbo calculus is a theory for genuine *operators* and their approximate solution operators, with many structural refinements; it has not been customary to study *relations* in this context. We therefore find it adequate to interpret the realisations of elliptic operators in terms of the theory based on [15], that characterises the elements in $\mathcal{M}$ by operators, rather than relations.

Let Ω be a smooth subset of $\mathbb{R}^n$, with C^∞ boundary $\partial\Omega = \Sigma$, let m be a positive integer, and let $A = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ be an m-th order differential operator on Ω with coefficients in $C^\infty(\overline{\Omega})$ and uniformly elliptic (i.e., the principal symbol $a^0(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is invertible for all $x \in \overline{\Omega}$, all $\xi \in \mathbb{R}^n \setminus \{0\}$). The maximal and minimal realisations in $H = L_2(\Omega)$ act like A in the distribution sense, with domains defined by

$$D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\}, \quad D(A_{\min}) = H_0^m(\Omega), \tag{3.1}$$

and it is well-known (from ellipticity arguments) that

$$A_{\min}^* = A'_{\max}, \quad A_{\max}^* = A'_{\min}, \tag{3.2}$$

where $A'_{\max}$ and $A'_{\min}$ are the analogous operators for the formal adjoint A' of A.

The operators belonging to $\mathcal{M}$ resp. $\mathcal{M}'$ defined as in Section 2.1 are called the *realisations* of A resp. A'.

We denote by $H^s(\Omega)$ the Sobolev space over Ω of order s, namely the space of restrictions to Ω of the elements of $H^s(\mathbb{R}^n)$, which consists of the distributions $u \in \mathcal{S}'$ such that $(1+|\xi|^2)^{s/2}\hat{u} \in L_2(\mathbb{R}^n)$. ($\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ is Schwartz' space of temperate distributions.) By $H_0^s(\overline{\Omega})$ we denote the subspace of $H^s(\mathbb{R}^n)$ of elements supported in $\overline{\Omega}$. For $s > -\frac{1}{2}$, $s - \frac{1}{2}$ not integer, this space can be identified with the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, also denoted $H_0^s(\Omega)$. Sobolev spaces over Σ, $H^s(\Sigma)$, are defined by use of local coordinates. We denote $\gamma_j u = (\partial_n^j u)|_\Sigma$, where ∂_n is the derivative along the interior normal $\vec{n}$ at Σ. Here γ_j maps $H^s(\Omega) \to H^{s-j-\frac{1}{2}}(\Sigma)$ for $j < s - \frac{1}{2}$. For $j < m$ there is an extension $\gamma_j : D(A_{\max}) \to H^{-j-\frac{1}{2}}(\Sigma)$, cf. e.g. Lions and Magenes [30].

Let us briefly recall the relevant elements of the ψdbo calculus. In its general form it treats operators

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{array}{c} C^\infty(\overline{\Omega})^N \\ \times \\ C^\infty(\Sigma)^M \end{array} \to \begin{array}{c} C^\infty(\overline{\Omega})^{N'} \\ \times \\ C^\infty(\Sigma)^{M'} \end{array}. \tag{3.3}$$

Here T is a generalized *trace operator*, going from Ω to Σ; K is a so-called *Poisson operator* (called a potential operator or coboundary operator in some other texts), going from Σ to Ω; S is a *pseudodifferential operator on* Σ; and G is an operator on Ω called a *singular Green operator*, a non-pseudodifferential term that has to be included in order to have adequate composition rules. P is a ψdo defined on an open set $\widetilde{\Omega} \supset \overline{\Omega}$, and P_+ is its truncation to Ω, defined by $P_+ u = r^+ P e^+ u$, where r^+ restricts $\mathcal{D}'(\widetilde{\Omega})$ to $\mathcal{D}'(\Omega)$, and e^+ extends locally integrable functions on $\overline{\Omega}$ by zero on $\widetilde{\Omega} \setminus \overline{\Omega}$. P is assumed to satisfy the so-called transmission condition at Σ; this holds for the operators derived from elliptic differential operators that we consider here. There are suitable Sobolev space mapping properties in terms of the orders of the entering operators.

A solvable elliptic PDE problem

$$Au = f \text{ on } \Omega, \quad Tu = \varphi \text{ on } \Sigma, \tag{3.4}$$

enters in this framework by an operator (where we suppress the index + on A since it acts locally)

$$\begin{pmatrix} A \\ T \end{pmatrix} : C^\infty(\overline{\Omega})^N \to \begin{array}{c} C^\infty(\overline{\Omega})^N \\ \times \\ C^\infty(\Sigma)^{M'} \end{array} \tag{3.5}$$

(note that $M = 0$ and $M' > 0$), with the inverse

$$\begin{pmatrix} R & K \end{pmatrix} : \begin{array}{c} C^\infty(\overline{\Omega})^N \\ \times \\ C^\infty(\Sigma)^{M'} \end{array} \to C^\infty(\overline{\Omega})^N; \quad R = Q_+ + G, \tag{3.6}$$

where $Q_+ + G$ solves the problem (3.4) with $\varphi = 0$ and K solves the problem (3.4) with $f = 0$. Here Q is a parametrix of A on $\widetilde{\Omega}$ (for example in case $A = -\Delta$, Q is the convolution with $c_n|x|^{2-n}$ on $\mathbb{R}^n$ when $n \geq 3$), and G is the correction term needed to make $Q_+ + G$ map into the functions satisfying the homogeneous boundary condition.

Besides providing a convenient terminology, the ψdbo calculus has the advantage that it gives complete composition rules: When $\mathcal{A}$ and $\mathcal{A}'$ are two systems as in (3.3), the composed operator $\mathcal{A}\mathcal{A}'$ again has this structure. In particular, a composition TK gives a ψdo on Σ, and a composition KT gives a singular Green operator. Compositions TP_+, TG and ST give trace operators, compositions P_+K, GK and KS give Poisson operators. These are the facts that we shall mainly use in the present paper. Details on the ψdbo calculus are found e.g. in [21], [22].

3.2 A typical second-order case

To give an impression of the theory, we begin by studying in some detail the case of a second-order strongly elliptic operator A. This part is divided into five subsections. In the first one we introduce boundary triplets for the operator A. In the next three subsections we concentrate on the case of "pure conditions", i.e. when $T : Z \to Z'$. For this case, we show in 3.2.2 how T can be identified with an operator L representing a Neumann-type boundary condition. In 3.2.3, we study the corresponding M-function, proving, among other things, a Kreĭn-type resolvent formula. Subsection 3.2.4 takes a closer look at problems with elliptic boundary conditions. Finally, in 3.2.5, we consider the general case when $T : V \to W$ and V, W are subspaces of Z and Z', respectively.

3.2.1 Boundary triplets

We begin by introducing boundary triplets for the case of a second-order strongly elliptic operator A i.e., with $\operatorname{Re} a^0(x,\xi) \ge c_0|\xi|^2$ for $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$ ($c_0 > 0$), taking Ω bounded. Let $s_0(x)$ be the (nonvanishing) coefficient of $-\partial_n^2$ when A is written in normal and tangential coordinates at a boundary point x, then A has the Green's formula

$$(Au, v)_{L_2(\Omega)} - (u, A'v)_{L_2(\Omega)} = (s_0\gamma_1 u, \gamma_0 v)_{L_2(\Sigma)} - (\gamma_0 u, \bar{s}_0\gamma_1 v + \mathcal{A}_0'\gamma_0 v)_{L_2(\Sigma)}, \tag{3.7}$$

for $u, v \in H^2(\Omega)$, with a suitable first-order differential operator $\mathcal{A}_0'$ over Σ. We denote $s_0\gamma_1 = \nu_1$, $\bar{s}_0\gamma_1 = \nu_1'$.

A simple example was explained in [7, Section 7], namely

$$A = -\Delta + p(x)\cdot \operatorname{grad}, \text{ with formal adjoint } A'v = -\Delta v - \operatorname{div}(\bar{p}v); \tag{3.8}$$

where p is an n-vector of functions in $C^\infty(\overline{\Omega})$.

We can assume, after addition of a constant to A if necessary, that the Dirichlet problem for A is uniquely solvable.

The Dirichlet realisation A_γ is the operator lying in $\mathcal{M}$ with domain

$$D(A_\gamma) = D(A_{\max}) \cap H_0^1(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$$

(the last equality follows by elliptic regularity theory); it has $0 \in \varrho(A_\gamma)$. Let

$$Z_\lambda^s(A) = \{u \in H^s(\Omega) \mid (A-\lambda)u = 0\}, \tag{3.9}$$

for $s \in \mathbb{R}$. It is known from [30] that the trace operators γ_0 and γ_1, hence also ν_1, extend by continuity to continuous maps

$$\gamma_0 : Z_\lambda^s(A) \to H^{s-\frac12}(\Sigma), \quad \gamma_1 \text{ and } \nu_1 : Z_\lambda^s(A) \to H^{s-\frac32}(\Sigma), \tag{3.10}$$

for all $s \in \mathbb{R}$. When $\lambda \in \varrho(A_\gamma)$, let $K_\gamma^\lambda : \varphi \mapsto u$ denote the Poisson operator solving the semi-homogeneous Dirichlet problem

$$(A-\lambda)u = 0 \text{ in } \Omega, \quad \gamma_0 u = \varphi. \tag{3.11}$$

It maps continuously

$$H^{s-\frac12}(\Sigma) \to H^s(\Omega), \text{ for all } s \in \mathbb{R}. \tag{3.12}$$

Moreover, it maps $H^{s-\frac12}(\Sigma)$ homeomorphically onto $Z_\lambda^s(A)$ for all $s \in \mathbb{R}$, with γ_0 acting as an inverse there. (We introduce below a special notation for the restricted operator when $s = 0$, see (3.21).) Analogously, there is a Poisson operator $K_\gamma'^{\bar\lambda}$ solving (3.11) with $A - \lambda$ replaced by $A' - \bar\lambda$, mapping $H^{s-\frac12}(\Sigma)$ homeomorphically onto $Z_{\bar\lambda}^s(A')$, with γ_0 acting as an inverse there.

Now define the Dirichlet-to-Neumann operators for each $\lambda \in \varrho(A_\gamma)$,

$$P_{\gamma_0,\nu_1}^\lambda = \nu_1 K_\gamma^\lambda; \quad P_{\gamma_0,\nu_1'}'^{\bar\lambda} = \nu_1' K_\gamma'^{\bar\lambda}; \tag{3.13}$$

they are a first-order elliptic pseudodifferential operators over Σ, continuous and Fredholm from $H^{s-\frac{1}{2}}(\Sigma)$ to $H^{s-\frac{3}{2}}(\Sigma)$ for all $s \in \mathbb{R}$ (details e.g. in [16]).

We shall use the notation for general trace maps β and η:

$$P^{\lambda}_{\beta,\eta} : \beta u \mapsto \eta u, \quad u \in Z^s_\lambda(A), \tag{3.14}$$

when this operator is well-defined.

Introduce the trace operators Γ and Γ' (from [15], where they were called M and M') by

$$\Gamma u = \nu_1 u - P^0_{\gamma_0,\nu_1}\gamma_0 u, \quad \Gamma' u = \nu'_1 u - P'^0_{\gamma_0,\nu'_1}\gamma_0 u. \tag{3.15}$$

Here Γ maps $D(A_{\max})$ continuously onto $H^{\frac{1}{2}}(\Sigma)$ and can also be written $\Gamma = \nu_1 A_\gamma^{-1} A_{\max}$, and Γ' has the analogous properties. Moreover, a generalised Green's formula is valid *for all* $u \in D(A_{\max})$, $v \in D(A'_{\max})$:

$$(Au, v)_{L_2(\Omega)} - (u, A'v)_{L_2(\Omega)} = (\Gamma u, \gamma_0 v)_{\frac{1}{2},-\frac{1}{2}} - (\gamma_0 u, \Gamma' v)_{-\frac{1}{2},\frac{1}{2}}, \tag{3.16}$$

where $(\cdot,\cdot)_{s,-s}$ denotes the duality pairing between $H^s(\Sigma)$ and $H^{-s}(\Sigma)$. Furthermore,

$$(Au, w)_{L_2(\Omega)} = (\Gamma u, \gamma_0 w)_{\frac{1}{2},-\frac{1}{2}} \text{ for all } w \in Z^0_0(A'). \tag{3.17}$$

(Cf. [15, Th. III 1.2].)

To achieve $L_2(\Sigma)$-dualities in the right-hand side of (3.16), one can choose the norms in $H^{\pm\frac{1}{2}}(\Sigma)$ to be induced by suitable isometries from the norm in $L_2(\Sigma)$. There exists a family of pseudodifferential elliptic invertible operators Λ_s of order $s \in \mathbb{R}$ on Σ, symmetric with respect to the duality in $L_2(\Sigma)$ and with $\Lambda_{-s} = \Lambda_s^{-1}$, such that when each $H^s(\Sigma)$ is provided with the norm for which Λ_s is an isometry from $H^s(\Sigma)$ onto $L_2(\Sigma)$, Λ_s also maps $H^t(\Sigma)$ isometrically onto $H^{t-s}(\Sigma)$, all t, and

$$(\Lambda_{-s}\varphi, \Lambda_s\psi)_{s,-s} = (\varphi,\psi)_{L_2(\Sigma)}, \quad \varphi,\psi \in L_2(\Sigma). \tag{3.18}$$

Then when we introduce composed operators

$$\Gamma_1 = \Lambda_{\frac{1}{2}}\Gamma, \quad \Gamma'_1 = \Lambda_{\frac{1}{2}}\Gamma', \quad \Gamma_0 = \Lambda_{-\frac{1}{2}}\gamma_0 = \Gamma'_0; \tag{3.19}$$

(3.16) takes the form (2.27) with $\mathcal{H} = \mathcal{K} = L_2(\Sigma)$:

Proposition 3.1 *For the adjoint pair* $A_{\min}$ *and* $A'_{\min}$, (3.19) *provides a boundary triplet with* $\mathcal{H} = \mathcal{K} = L_2(\Sigma)$*:*

$$(Au, v)_{L_2(\Omega)} - (u, A'v)_{L_2(\Omega)} = (\Gamma_1 u, \Gamma'_0 v)_{L_2(\Sigma)} - (\Gamma_0 u, \Gamma'_1 v)_{L_2(\Sigma)}, \tag{3.20}$$

holds when $u \in D(A_{\max})$, $v \in D(A'_{\max})$*:*

Such reductions to L_2-dualities are made in [7] and [37]. [15] did not make the modification by composition with $\Lambda_{\pm\frac{1}{2}}$, but worked directly with (3.16). (This was in order to avoid introducing too many operators. Another reason was that the Sobolev spaces $H^s(\Sigma)$ do not have a "preferred norm" when $s \neq 0$; only the duality $(\cdot,\cdot)_{s,-s}$ should be consistent with the self-duality of $L_2(\Sigma)$. Moreover, when the realisation $\widetilde{A}$ represents an *elliptic* boundary condition, $D(\widetilde{A}) \subset H^2(\Omega)$ and the boundary values are in $L_2(\Sigma)$. — Various homeomorphisms were used in [17] for the sake of numerical comparison.)

In the rest of this section, *we use the abbreviation* H^s *for* $H^s(\Sigma)$. We shall keep the formulation with dualities in the study of pure Neumann-type boundary conditions, but return to (3.19) in connection with more general boundary conditions.

3.2.2 Interpretation of the boundary conditions

Consider the set-up of Section 2.1 with $A_\beta = A_\gamma$, the projection pr_β being denoted pr_γ. The realisation A_γ itself of course corresponds to the case $V = W = \{0\}$ in Theorem 2.1.

Let $\widetilde{A}$ be a closed realisation which corresponds to an operator T with $V = Z$, $W = Z'$ by Theorem 2.1. Note that

$$Z = Z_0^0(A),\ Z' = Z_0^0(A'), \text{ and for } \lambda \in \varrho(A_\gamma),\ Z_\lambda = Z_\lambda^0(A),\ Z'_{\bar\lambda} = Z_{\bar\lambda}^0(A');$$

closed subspaces of $L_2(\Omega)$.

Denote the restrictions of γ_0 to mappings from Z_λ resp. $Z'_{\bar\lambda}$ to $H^{-\frac{1}{2}}$ by γ_{Z_λ} resp. $\gamma_{Z'_{\bar\lambda}}$; they are homeomorphisms

$$\gamma_{Z_\lambda} : Z_\lambda \xrightarrow{\sim} H^{-\frac{1}{2}}, \quad \gamma_{Z'_{\bar\lambda}} : Z'_{\bar\lambda} \xrightarrow{\sim} H^{-\frac{1}{2}}, \tag{3.21}$$

and their inverses $\gamma_{Z_\lambda}^{-1}$ resp. $\gamma_{Z'_{\bar\lambda}}^{-1}$ coincide with K_γ^λ resp. $K'^{\bar\lambda}_\gamma$ but have the restricted range space. Their adjoints map

$$\gamma_{Z_\lambda}^* : H^{\frac{1}{2}} \xrightarrow{\sim} Z_\lambda, \quad \gamma_{Z'_{\bar\lambda}}^* : H^{\frac{1}{2}} \xrightarrow{\sim} Z'_{\bar\lambda}.$$

When $\lambda = 0$, the λ-indications are left out.

We shall first interpret $\widetilde{A}$ in terms of a boundary condition using the maps with $\lambda = 0$; this stems from [15]. The above homeomorphisms allow "translating" an operator $T : Z \to Z'$ to an operator $L : H^{-\frac{1}{2}} \to H^{\frac{1}{2}}$, as in the diagram

$$\begin{tikzcd} Z \arrow[r, "\gamma_Z"] \arrow[d, "T"'] & H^{-\frac{1}{2}} \arrow[d, "L"] \\ Z' \arrow[r, "(\gamma_{Z'}^*)^{-1}"'] & H^{\frac{1}{2}} \end{tikzcd} \qquad D(L) = \gamma_Z D(T), \tag{3.22}$$

where the horizontal maps are homeomorphisms. In other words,

$$L = (\gamma_{Z'}^*)^{-1} T \gamma_Z^{-1}, \text{ with } D(L) = \gamma_Z D(T) \tag{3.23}$$

(a closed densely defined operator from $H^{-\frac{1}{2}}$ to $H^{\frac{1}{2}}$). Hereby we have, when $\varphi = \gamma_Z z \in D(L)$, $\psi = \gamma_{Z'} w \in H^{-\frac{1}{2}}$,

$$(Tz, w)_{Z'} = (T\gamma_Z^{-1}\varphi, \gamma_{Z'}^{-1}\psi)_{Z'} = (L\varphi, \psi)_{\frac{1}{2},-\frac{1}{2}}.$$

Note that $D(L) = \gamma_0 D(T) = \gamma_0 \operatorname{pr}_\zeta D(\widetilde{A}) = \gamma_0 D(\widetilde{A})$, since γ_0 vanishes on $D(A_\gamma)$.

Recall the equation defining T from $\widetilde{A}$:

$$(T \operatorname{pr}_\zeta u, w) = (Au, w) \text{ for } u \in D(\widetilde{A}),\ w \in Z'. \tag{3.24}$$

In view of (3.17), the right-hand side may be written

$$(Au, w) = (\Gamma u, \gamma_0 w)_{\frac{1}{2},-\frac{1}{2}} = (\nu_1 u - P_{\gamma_0,\nu_1}\gamma_0 u, \gamma_0 w)_{\frac{1}{2},-\frac{1}{2}} \text{ for all } w \in Z'. \tag{3.25}$$

For the left-hand side we have with L defined above, using that $\gamma_Z \operatorname{pr}_\zeta u = \gamma_0(u - \operatorname{pr}_\gamma u) = \gamma_0 u$,

$$(T \operatorname{pr}_\zeta u, w) = (L\gamma_Z \operatorname{pr}_\zeta u, \gamma_{Z'} w)_{\frac{1}{2},-\frac{1}{2}} = (L\gamma_0 u, \gamma_0 w)_{\frac{1}{2},-\frac{1}{2}}.$$

Then, when we write $\gamma_0 w = \psi$, (3.24) takes the form

$$(L\gamma_0 u, \psi)_{\frac{1}{2},-\frac{1}{2}} = (\nu_1 u - P^0_{\gamma_0,\nu_1}\gamma_0 u, \psi)_{\frac{1}{2},-\frac{1}{2}} \text{ for all } u \in D(\widetilde{A}),\ \text{all } \psi \in H^{-\frac{1}{2}}. \tag{3.26}$$

Since ψ runs through $H^{-\frac12}$, this may be written $L\gamma_0 u = \nu_1 u - P^0_{\gamma_0,\nu_1}\gamma_0 u$, or,

$$\nu_1 u = (L + P^0_{\gamma_0,\nu_1})\gamma_0 u, \quad \gamma_0 u \in D(L). \tag{3.27}$$

So in fact $\widetilde{A}$ represents a Neumann-type boundary condition (3.27).

Conversely, if we want $\widetilde{A}$ to represent a given Neumann-type boundary condition

$$\nu_1 u = C\gamma_0 u, \tag{3.28}$$

where C is a ψdo over Σ, we see that L has to be taken to act like

$$L = C - P^0_{\gamma_0,\nu_1}. \tag{3.29}$$

Now let us turn to the λ-dependent case. Here we consider the families $\widetilde{A} - \lambda$ and T^λ and can proceed in a very similar way. When working with the concrete boundary Sobolev spaces we find the advantage that Z and Z_λ are mapped by γ_0 to the same space $H^{-\frac12}$. In fact,

$$\gamma_{Z_\lambda} = \gamma_Z F^\lambda_Z, \quad \gamma_Z = \gamma_{Z_\lambda} E^\lambda_Z, \text{ and similarly } \gamma_{Z'_{\bar\lambda}} = \gamma_{Z'} F'^{\bar\lambda}_{Z'}, \quad \gamma_{Z'} = \gamma_{Z'_{\bar\lambda}} E'^{\bar\lambda}_{Z'}, \tag{3.30}$$

since, e.g., $\gamma_0 F^\lambda u = \gamma_0(u - \lambda A_\gamma^{-1} u) = \gamma_0 u$, cf. (2.16).

Let $\lambda \in \varrho(A_\gamma)$. In the defining equation for T^λ,

$$(T^\lambda \operatorname{pr}^\lambda_\zeta u, w) = ((A-\lambda)u, w) \text{ for } u \in D(\widetilde{A}), w \in Z'_{\bar\lambda}, \tag{3.31}$$

we rewrite the two sides as

$$\begin{aligned} (T^\lambda u^\lambda_\zeta, w) &= (L^\lambda \gamma_{Z_\lambda} u^\lambda_\zeta, \gamma_{Z'_{\bar\lambda}} w)_{\frac12,-\frac12}, \\ ((A-\lambda)u, w) &= (\nu_1 u - P^\lambda_{\gamma_0,\nu_1}\gamma_0 u, \gamma_0 w)_{\frac12,-\frac12}, \end{aligned}$$

where $L^\lambda : H^{-\frac12} \to H^{\frac12}$ is defined by

$$L^\lambda = (\gamma^*_{Z'_{\bar\lambda}})^{-1} T^\lambda \gamma^{-1}_{Z_\lambda}, \quad D(L^\lambda) = \gamma_{Z_\lambda} D(T^\lambda); \tag{3.32}$$

note that

$$D(L^\lambda) = \gamma_0 D(T^\lambda) = \gamma_0 E^\lambda D(T) = \gamma_0 D(T) = D(L). \tag{3.33}$$

Then since $\gamma_{Z_\lambda} u^\lambda_\zeta = \gamma_0 u$, the operator $\widetilde{A} - \lambda$ represents the boundary condition

$$L^\lambda \gamma_0 u = \nu_1 u - P^\lambda_{\gamma_0,\nu_1}\gamma_0 u, \quad \gamma_0 u \in D(L^\lambda) = D(L). \tag{3.34}$$

Moreover, in view of Corollary 2.5, L^λ is related to $T + G^\lambda$ as follows:

$$\begin{tikzcd} Z \arrow[r, "E^\lambda_Z"] \arrow[d, "T+G^\lambda"'] & Z_\lambda \arrow[r, "\gamma_{Z_\lambda}"] \arrow[d, "T^\lambda"'] & H^{-\frac12} \arrow[d, "L^\lambda"] \\ Z' \arrow[r, "(F'^{\bar\lambda}_{Z'})^*"'] & Z'_{\bar\lambda} \arrow[r, "(\gamma^*_{Z'_{\bar\lambda}})^{-1}"'] & H^{\frac12} \end{tikzcd}$$

where the horizontal maps are homeomorphisms. In view of (3.30), they compose to γ_Z resp. $(\gamma^*_{Z'})^{-1}$, so

$$L^\lambda = (\gamma^*_{Z'})^{-1}(T + G^\lambda)\gamma_Z^{-1} = L + (\gamma^*_{Z'})^{-1} G^\lambda \gamma_Z^{-1}. \tag{3.35}$$

Since $D(\widetilde{A} - \lambda) = D(\widetilde{A})$, (3.34) and (3.27) define the same boundary condition, hence

$$L^\lambda = L + P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1} \text{ on } D(L). \tag{3.36}$$

Remark 3.2 In particular, it can be inferred (e.g. from the case $L = 0$) that $(\gamma^*_{Z'})^{-1} G^\lambda \gamma_Z^{-1} = P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1}$. Note how the operator family L^λ (in this case where $V = Z$, $W = Z'$) is written as the sum of a λ-independent operator L (defining the domain of the realisation) and a λ-dependent operator $P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1}$, which is universal in the sense that it only depends on A, the set Ω, and λ. It is useful to observe that since G^λ is continuous from Z to Z' for each λ, $P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1}$ is continuous from $H^{-\frac12}$ to $H^{\frac12}$, hence is of order -1, in contrast to its two individual terms that are elliptic of order 1 (having the same principal symbol).

This analysis results in the theorem:

Theorem 3.3 *For the second-order strongly elliptic operator A introduced above, let $\widetilde{A}$ be a closed realisation with $\operatorname{pr}_\zeta D(\widetilde{A})$ dense in Z and $\operatorname{pr}_{\zeta'} D(\widetilde{A}^*)$ dense in Z'. Let $T : Z \to Z'$ be the operator it corresponds to by Theorem* 2.1.

(i) *When Z and Z' are mapped to $H^{-\frac12}$ by γ_0 and Theorem* 2.1 *is carried over to the setting based on the Green's formula* (3.16), *$\widetilde{A}$ corresponds to a closed, densely defined operator $L : H^{-\frac12} \to H^{\frac12}$ with domain $D(L) = \gamma_0 D(\widetilde{A})$ such that $\widetilde{A}$ represents the boundary condition*

$$\nu_1 u = C\gamma_0 u, \text{ where } C = L + P^0_{\gamma_0,\nu_1}. \tag{3.37}$$

Here L is defined from T by (3.23)*.*

(ii) *For any $\lambda \in \varrho(A_\gamma)$, $\widetilde{A} - \lambda$ corresponds similarly to*

$$L^\lambda = L + P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1} : H^{-\frac12} \to H^{\frac12}, \tag{3.38}$$

with domain $D(L^\lambda) = D(L)$. $\widetilde{A} - \lambda$ is the realisation of $A - \lambda$ determined by the boundary condition (3.37)*, where C may also be written $C = L^\lambda + P^\lambda_{\gamma_0,\nu_1}$. Here, when $\widetilde{A} - \lambda$ corresponds to T^λ by Corollary* 2.3*, L^λ is defined from T^λ by* (3.32)*. Moreover,* (3.35) *holds.*

(iii) *Furthermore, for any $\lambda \in \varrho(A_\gamma)$,*

$$\begin{aligned} \ker(\widetilde{A} - \lambda) &= K^\lambda_\gamma \ker L^\lambda, \\ \operatorname{ran}(\widetilde{A} - \lambda) &= \gamma^*_{Z'_{\bar\lambda}} \operatorname{ran} L^\lambda + \operatorname{ran}(A_{\min} - \lambda). \end{aligned} \tag{3.39}$$

Proof. All has been accounted for above except point (iii), but this follows immediately from (2.14). □

3.2.3 The M-function

We define an M-function in this representation, by use of Lemma 2.8. We have from (2.32) for $\lambda \in \varrho(\widetilde{A})$:

$$M_{\widetilde{A}}(\lambda) = \operatorname{pr}_\zeta\big((I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda)\big) A_\gamma^{-1} \mathrm{i}_{Z' \to H} : Z' \to Z. \tag{3.40}$$

We know from Theorem 2.9 that the M-function should coincide with minus the inverse of the operator induced from T^λ in (2.36), when $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$. So, applying the trace maps in (3.21) in a similar way as we did for T, we get

$$\begin{aligned} M_L(\lambda) &= \gamma_Z M_{\widetilde{A}}(\lambda) \gamma^*_{Z'} = \gamma_Z \operatorname{pr}_\zeta\big((I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda)\big) A_\gamma^{-1} \mathrm{i}_{Z' \to H}\, \gamma^*_{Z'} \\ &= \gamma_0\big(I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda)\big) A_\gamma^{-1} \mathrm{i}_{Z' \to H}\, \gamma^*_{Z'}. \end{aligned} \tag{3.41}$$

Here, when $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$, $-M_L(\lambda)$ is the inverse of the operator translated from $T + G^\lambda$, namely, in view of (3.35)–(3.36),

$$M_L(\lambda) = -(L + P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1})^{-1} = -(L^\lambda)^{-1}, \tag{3.42}$$

bounded from $H^{\frac12}$ to $H^{-\frac12}$, and then it has range $D(L)$. Moreover, it has the extension to $\lambda \in \varrho(\widetilde{A})$ given in (3.41), a holomorphic family of bounded operators from $H^{\frac12}$ to $H^{-\frac12}$.

Note that when $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$, L^λ is surjective onto $H^{\frac12}$. These considerations lead to:

Theorem 3.4 *For the realisation considered in Theorem* 3.3*, there is an M-function defined by*

$$M_L(\lambda) = \gamma_0\big(I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda)\big)A_\gamma^{-1}\,\mathrm{i}_{Z'\to H}\,\gamma_{Z'}^*, \tag{3.43}$$

a holomorphic family of bounded operators from $H^{\frac12}$ to $H^{-\frac12}$. For $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$, it satisfies

$$M_L(\lambda) = -(L + P^0_{\gamma_0,\gamma_1} - P^\lambda_{\gamma_0,\gamma_1})^{-1} = -(L^\lambda)^{-1}. \tag{3.44}$$

There is the following Kreĭn resolvent formula, valid for all $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$:

$$\begin{aligned}(\widetilde{A} - \lambda)^{-1} &= (A_\gamma - \lambda)^{-1} - \mathrm{i}_{Z_\lambda\to H}\,\gamma_{Z_\lambda}^{-1} M_L(\lambda)(\gamma^*_{Z'_{\bar\lambda}})^{-1}\,\mathrm{pr}_{Z'_{\bar\lambda}} \\ &= (A_\gamma - \lambda)^{-1} - K^\lambda_\gamma M_L(\lambda)(K'^{\bar\lambda}_\gamma)^*.\end{aligned} \tag{3.45}$$

P r o o f. The definition of M_L is accounted for above. The first line in the Kreĭn formula follows from (2.42) in the case $V = Z$, $W = Z'$, by the calculation

$$E^\lambda_Z M_{\widetilde{A}}(\lambda)(E'^{\bar\lambda}_{Z'})^* = E^\lambda_Z \gamma_Z^{-1} M_L(\lambda)(\gamma^*_{Z'})^{-1}(E'^{\bar\lambda}_{Z'})^* = \gamma^{-1}_{Z_\lambda} M_L(\lambda)(\gamma^*_{Z'_{\bar\lambda}})^{-1},$$

using (3.41) and (3.30). The second line follows since $\mathrm{i}_{Z_\lambda\to H}\,\gamma^{-1}_{Z_\lambda} = K^\lambda_\gamma : H^{-\frac12} \to H$, $(\gamma^*_{Z'_{\bar\lambda}})^{-1}\,\mathrm{pr}_{Z'_{\bar\lambda}} = (\mathrm{i}_{Z'_{\bar\lambda}\to H}\,\gamma^{-1}_{Z'_{\bar\lambda}})^* = (K'^{\bar\lambda}_\gamma)^* : H \to H^{\frac12}$ (recall that $H = L_2(\Omega)$). □

Note that with the notation (3.14), $L^\lambda = -P^\lambda_{\gamma_0,\nu_1 - C\gamma_0}$ and $M_L(\lambda) = P^\lambda_{\nu_1 - C\gamma_0,\gamma_0}$, cf. (3.29).

3.2.4 Elliptic boundary conditions

Further information can be obtained in elliptic cases. Using the Sobolev space mapping properties of γ_Z and its inverse, one finds since $D(A_\gamma) \subset H^2(\Omega)$ that $D(\widetilde{A}) \subset H^2(\Omega)$ if and only if $D(L) \subset H^{\frac32}$. If L is given as an arbitrary ψdo of order 1, it is not in general bounded from $H^{-\frac12}$ to $H^{\frac12}$, but has a subset as domain.

Ellipticity of the boundary value problem defined from (3.28), (3.29) (the Shapiro-Lopatinskiĭ condition) holds precisely when L acts like an *elliptic* ψdo of order 1. Then $\operatorname{ran} L \subset H^{\frac12}$ implies $D(L) \subset H^{\frac32}$, and the graph-norm on $D(L)$ is equivalent with the $H^{\frac32}$-norm. Let $\varrho(\widetilde{A}) \cap \varrho(A_\gamma) \neq \emptyset$ (a reasonable hypothesis in our discussion) and let $\lambda_0 \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$. Then L^{λ_0} has kernel and cokernel $\{0\}$ (since T^{λ_0} has so, by Theorem 2.1) and is likewise elliptic of order 1, so the inverse $-M_L(\lambda_0)$ is an elliptic ψdo of order -1; it is defined on all of $H^{\frac12}$. It maps $H^{\frac12}$ onto $H^{\frac32}$ (since the range of the operator from any H^s to H^{s+1} has codimension 0). It follows that

$$D(L) = D(L^{\lambda_0}) = H^{\frac32}. \tag{3.46}$$

Then moreover, $D(\widetilde{A}) \subset H^2(\Omega)$ (and $D(\widetilde{A})$ is the largest subset of $D(A_{\max})$ where (3.28) holds).

In this case, the adjoint of $(L^{\lambda_0})^{-1}$ (as a bounded operator from $H^{\frac12}$ to $H^{-\frac12}$) is $((L^{\lambda_0})^*)^{-1}$, so the adjoint of L (as an unbounded closed densely defined operator from $H^{-\frac12}$ to $H^{\frac12}$) is L^* with domain $H^{\frac32}$. Then $\widetilde{A}^*$ is the realisation of A' defined by the elliptic boundary condition

$$\nu_1' u = (L^* + P'^0_{\gamma_0,\nu_1'})\gamma_0 u, \tag{3.47}$$

and $D(\widetilde{A}^*) \subset H^2(\Omega)$. Note that if $A = A'$ and L is elliptic of order 1 and symmetric, $\widetilde{A}$ is selfadjoint.

Before including these observations in a theorem we shall show that also pr_{Z_λ}, $\mathrm{pr}_{Z'_{\bar\lambda}}$ and their adjoints $\mathrm{i}_{Z_\lambda\to H}$, $\mathrm{i}_{Z'_{\bar\lambda}\to H}$, belong to the ψdbo calculus; this will allow a discussion of $M_L(\lambda)$ for $\lambda \in \varrho(\widetilde{A}) \setminus \varrho(A_\gamma)$.

Proposition 3.5 *Let $\lambda \in \varrho(A_\gamma)$. Then $\mathrm{pr}_{Z'_{\bar\lambda}}$ acts as the singular Green operator*

$$\mathrm{pr}_{Z'_{\bar\lambda}} = I - (A_{\max} - \lambda)R((A' - \bar\lambda)(A - \lambda), \gamma_0, \gamma_1)(A'_{\max} - \bar\lambda), \tag{3.48}$$

where $R((A'-\bar\lambda)(A-\lambda),\gamma_0,\gamma_1): g\mapsto u$ *is the solution operator for the problem*

$$(A'-\bar\lambda)(A-\lambda)u=g,\quad \gamma_0u=\gamma_1u=0. \tag{3.49}$$

Similarly,

$$\operatorname{pr}_{Z_\lambda}=I-(A'_{\max}-\bar\lambda)R((A-\lambda)(A'-\bar\lambda),\gamma_0,\gamma_1)(A_{\max}-\lambda). \tag{3.50}$$

Moreover, $\mathrm{i}_{Z_\lambda\to H}$ *acts like the adjoint of the operator in* (3.50) *in the* ψ*dbo calculus, and* $\mathrm{i}_{Z'_{\bar\lambda}\to H}$ *acts like the adjoint of the operator in* (3.48)*.*

P r o o f. When $f\in H$, $\operatorname{pr}_{Z'_{\bar\lambda}}f$ is the second component of f in the decomposition

$$H=\operatorname{ran}(A_{\min}-\lambda)\oplus Z'_{\bar\lambda}.$$

Write $f=f_1+f_2$ according to this decomposition, and note that

$$(A'-\bar\lambda)f=(A'-\bar\lambda)f_1. \tag{3.51}$$

Observe moreover, that $v=(A_\gamma-\lambda)^{-1}f_1$ is the unique element in $D(A_{\min})=H^2_0(\Omega)$ such that $(A-\lambda)v=f_1$. In view of (3.51), v moreover solves

$$(A'-\bar\lambda)(A-\lambda)v=(A'-\bar\lambda)f,\quad \gamma_0v=\gamma_1v=0, \tag{3.52}$$

and this solution is unique since $(A'-\bar\lambda)(A-\lambda)$ is formally selfadjoint strongly elliptic with positive minimal realisation, hence has a positive Friedrichs extension, representing its Dirichlet problem (3.49). Thus v is uniquely determined as

$$v=R((A'-\bar\lambda)(A-\lambda),\gamma_0,\gamma_1)(A'-\bar\lambda)f,$$

and f_2 is given by the formula (3.48). It should be noted that the operator in (3.48) has a good meaning on $L_2(\Omega)$ in the ψdbo calculus; first $A'-\lambda$ maps $L_2(\Omega)$ continuously into $H^{-2}(\Omega)$, then $R((A'-\bar\lambda)(A-\lambda),\gamma_0,\gamma_1)$ maps $H^{-2}(\Omega)$ homeomorphically onto $H^2_0(\Omega)$, as is known for Dirichlet problems for positive operators, and finally $A-\lambda$ maps $H^2_0(\Omega)$ continuously into $L_2(\Omega)$. (3.48) defines a singular Green operator since the ψdo part of the second term cancels out with I.

The formula (3.50) follows by interchanging the roles of $A-\lambda$ and $A'-\bar\lambda$.

Finally, as a technical point taken care of in [20], the operator in (3.48), although the factor to the right is of order 2, is of class 0 since $R((A'-\bar\lambda)(A-\lambda),\gamma_0,\gamma_1)$ is of class -2. Then it does have an adjoint in the ψdbo calculus, so the assertion follows since the adjoint of $\operatorname{pr}_{Z_\lambda}: H\to Z_\lambda$ is $\mathrm{i}_{Z_\lambda\to H}$. □

Theorem 3.6 *For the operators considered in Theorems* 3.3 *and* 3.4*, one has:*

(i) *When* L *acts like an elliptic* ψ*do of order* 1*, then* $D(L)\subset H^{\frac32}$ *and* $D(\widetilde A)\subset H^2(\Omega)$*, and when* $\lambda\in\varrho(\widetilde A)\cap\varrho(A_\gamma)$*, all the operators entering in the formulas belong to the* ψ*dbo calculus, and* $M_L(\lambda)$ *is elliptic of order* -1*. Moreover, if* $\varrho(\widetilde A)\cap\varrho(A_\gamma)\neq\emptyset$*,* $D(L)=H^{\frac32}$*.*

(ii) *When the statements in* (i) *hold, we have moreover that* $\widetilde A^*$ *is the realisation of* A' *determined by the elliptic boundary condition* (3.47)*, and* $D(L^*)=H^{\frac32}$*,* $D(\widetilde A^*)\subset H^2(\Omega)$*. In particular, if* $A=A'$ *and* L *is symmetric and elliptic of order* 1*,* $\widetilde A$ *is selfadjoint with* $D(\widetilde A)\subset H^2(\Omega)$*.*

(iii) *When* L *is elliptic of order* 1 *with* $D(L)=H^{\frac32}$*,* $M_L(\lambda)$ *is an elliptic* ψ*do of order* -1 *for all* $\lambda\in\varrho(\widetilde A)$*.*

P r o o f. The assertions in (i) and (ii) were shown above. For (iii), we use the formula (3.43) derived in (3.41) from (2.32). When L is elliptic with $D(L)=H^{\frac32}$, the resolvent $(\widetilde A-\lambda)^{-1}$ belongs to the ψdbo calculus and maps $L_2(\Omega)$ into $H^2(\Omega)$; then the composition rules in the ψdbo calculus imply that $M_L(\lambda)$ is a ψdo over Σ. Since $\big(I-(\widetilde A-\lambda)^{-1}(A_{\max}-\lambda)\big)A_\gamma^{-1}$ maps $L_2(\Omega)$ continuously into $H^2(\Omega)$, $M_L(\lambda)$ maps $H^{\frac12}$ continuously into $H^{\frac32}$, hence is of order -1, for all $\lambda\in\varrho(\widetilde A)$. Since the *principal* symbol of $M_{\widetilde A}(\lambda)$ (in the ψdbo calculus) is independent of λ, also the principal symbol of the ψdo $M_L(\lambda)$ is independent of λ; it equals the inverse of the principal symbol of L and is therefore elliptic. □

Remark 3.7 The Kreĭn resolvent formula (3.45) can be compared with the standard resolvent formula $(\widetilde{A}-\lambda)^{-1} = Q_{\lambda,+} + G_\lambda$ (cf. Seeley [40] and e.g. Grubb [21]), where Q_λ is a parametrix (an approximate inverse) of $A-\lambda$ on a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$, $Q_{\lambda,+}$ is its truncation to Ω, and G_λ is a singular Green operator adapted to the boundary condition (as in (3.6)). This has been used e.g. to show asymptotic kernel and trace expansions (also for the associated heat operator). Formula (3.45) gives more direct eigenvalue information.

Remark 3.8 The operators L and $M_L(\lambda)$ can be pseudodifferential also in non-elliptic cases. A striking case is where $L=0$ as an operator from $H^{-\frac12}$ to $H^{\frac12}$, so that $\widetilde{A}$ represents the boundary condition

$$\nu_1 u = P^0_{\gamma_0,\nu_1}\gamma_0 u. \tag{3.53}$$

This is Kreĭn's "soft extension" [26], namely the realisation A_M with domain

$$D(A_M) = D(A_{\min}) \dot{+} Z, \tag{3.54}$$

as described in [15]. Its domain is not contained in any $H^s(\Omega)$ with $s>0$. Here $M_L(\lambda) = -(P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1})^{-1}$ when this inverse exists; cf. (3.44). Spectral properties of A_M are worked out in [19].

3.2.5 The general case

The previous subsections cover the cases where T goes from Z to Z' in Theorem 2.1 (called "pure conditions" in [15]). The family $\mathcal{M}$ moreover contains the operators corresponding to closed, densely defined operators $T: V\to W$ with arbitrary closed subspaces $V\subset Z$, $W\subset Z'$. In [15], it was shown that these correspond to operators $L: X\to Y^*$, $X=\gamma_0 V$ and $Y=\gamma_0 W$ (subspaces of $H^{-\frac12}$); then $\widetilde{A}$ represents the boundary condition

$$\gamma_0 u\in X, \quad (L\gamma_0 u,\psi)_{Y^*,Y} = (\Gamma u,\psi)_{\frac12,-\frac12}, \text{ all } \psi\in Y. \tag{3.55}$$

The equation is also written $L\gamma_0 u = \Gamma u|_Y$ (restriction as a functional on Y).

Let us show how this looks when we use the modified trace operators in (3.19) mapping the maximal domains to $L_2(\Sigma)$. Setting

$$X_1 = \Gamma_0 V = \Lambda_{-\frac12}X,\ Y_1 = \Gamma_0 W = \Lambda_{-\frac12}Y, \quad L_1 = (\Gamma^*_{0,W})^{-1}T\Gamma^{-1}_{0,V}, \text{ with } D(L_1) = \Gamma_0 D(T), \tag{3.56}$$

where $\Gamma_{0,V}$ resp. $\Gamma_{0,W}$ denote the restrictions of Γ_0 as mappings from V to X_1 resp. from W to Y_1, we have the following diagram:

$$\begin{tikzcd} V \arrow[r, "\Gamma_{0,V}"] \arrow[d, "T"'] & X_1 \arrow[d, "L_1"] \\ W \arrow[r, "(\Gamma^*_{0,W})^{-1}"'] & Y_1 \end{tikzcd} \tag{3.57}$$

We find as in (3.25)–(3.27) that the statements $\mathrm{pr}_\zeta\, u\in D(T)$, $Tu_\zeta = (Au)_W$, carry over to the statements

$$\Gamma_0 u\in D(L_1)\subset X_1, \quad L_1\Gamma_0 u = \mathrm{pr}_{Y_1}\Gamma_1 u; \tag{3.58}$$

this is then the boundary condition represented by $\widetilde{A}$. Note that the condition $\Gamma_0 u\in X_1$ enters as an important part of the boundary condition, compensating for the fact that L_1 acts between smaller spaces than in the case $V=Z, W=Z'$.

Since $\mathrm{pr}_{Y_1}\Gamma_1 u = \mathrm{pr}_{Y_1}\Lambda_{\frac12}(\nu_1 u - P^0_{\gamma_0,\nu_1}\gamma_0 u)$, (3.58) may also be written in terms of the standard traces $\gamma_0 u$ and $\nu_1 u$, as

$$\Lambda_{-\frac12}\gamma_0 u\in D(L_1), \quad \mathrm{pr}_{Y_1}\Lambda_{\frac12}\nu_1 u = (L_1 + \mathrm{pr}_{Y_1}\Lambda_{\frac12}P^0_{\gamma_0,\nu_1}\Lambda_{\frac12}\,\mathrm{i}_{X_1\to L_2})\Lambda_{-\frac12}\gamma_0 u. \tag{3.59}$$

Such formulations can likewise be pursued for $\widetilde{A}-\lambda$, allowing an extension of Theorems 3.3 and 3.4. Let $\lambda \in \varrho(A_\gamma)$. Now $\Gamma_{0,V_\lambda} = \Gamma_{0,V} F_V^\lambda : V_\lambda \xrightarrow{\sim} X_1$, and $\Gamma_{0,W_{\bar\lambda}} = \Gamma_{0,W} F_W'^{\bar\lambda} : W_{\bar\lambda} \xrightarrow{\sim} Y_1$. The defining equation for T^λ is:

$$(T^\lambda \operatorname{pr}_\zeta^\lambda u, w) = ((A-\lambda)u, w) \text{ for } u \in D(\widetilde{A}), w \in W_{\bar\lambda}. \tag{3.60}$$

We define

$$L_1^\lambda = (\Gamma_{0,W_{\bar\lambda}}^*)^{-1} T^\lambda \Gamma_{0,V_\lambda}^{-1}, \text{ with } D(L_1^\lambda) = \Gamma_{0,V_\lambda} D(T^\lambda) = \Gamma_{0,V} D(T) = D(L_1), \tag{3.61}$$

and then rewrite the two sides as follows, denoting $\Gamma_1^\lambda = \Lambda_{\frac12}(\nu_1 - P^\lambda_{\gamma_0,\nu_1}\gamma_0)$:

$$\begin{aligned} (T^\lambda u_\zeta^\lambda, w) &= (L_1^\lambda \Gamma_{0,V_\lambda} u_\zeta^\lambda, \Gamma_{0,W_{\bar\lambda}} w)_{L_2(\Sigma)} = (L_1^\lambda \Gamma_0 u, \Gamma_0 w)_{Y_1}, \\ ((A-\lambda)u, w) &= (\Gamma_1^\lambda u, \Gamma_0 w)_{L_2(\Sigma)} = (\operatorname{pr}_{Y_1} \Gamma_1^\lambda u, \Gamma_0 w)_{Y_1}. \end{aligned}$$

When w runs through $W_{\bar\lambda}$, $\Gamma_0 w$ runs through Y_1, so we see that $\widetilde{A}-\lambda$ represents the boundary condition

$$\Gamma_0 u \in D(L_1) \subset X_1, \quad L_1^\lambda \Gamma_0 u = \operatorname{pr}_{Y_1} \Gamma_1^\lambda u. \tag{3.62}$$

It can also be written in terms of the standard trace maps as

$$\Lambda_{-\frac12}\gamma_0 u \in D(L_1) \subset X_1, \quad L_1^\lambda \Lambda_{-\frac12}\gamma_0 u = \operatorname{pr}_{Y_1} \Lambda_{\frac12}(\nu_1 u - P^\lambda_{\gamma_0,\nu_1}\gamma_0 u), \tag{3.63}$$

where the equation can be rewritten as

$$\operatorname{pr}_{Y_1} \Lambda_{\frac12}\nu_1 u = (L_1^\lambda + \operatorname{pr}_{Y_1} \Lambda_{\frac12} P^\lambda_{\gamma_0,\nu_1} \Lambda_{\frac12} \operatorname{i}_{X_1\to L_2}) \Lambda_{-\frac12}\gamma_0 u. \tag{3.64}$$

Since $D(\widetilde{A}-\lambda) = D(\widetilde{A})$, we have in view of (3.59):

$$L_1^\lambda = L_1 + \operatorname{pr}_{Y_1} \Lambda_{\frac12}(P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1}) \Lambda_{\frac12} \operatorname{i}_{X_1\to L_2}, \text{ when } \lambda \in \varrho(A_\gamma). \tag{3.65}$$

All this leads to:

Theorem 3.9 *For the second-order strongly elliptic operator A introduced above, let the closed realisation $\widetilde{A}$ correspond to $T : V \to W$ by Theorem* 2.1.

(i) *Define X_1, Y_1 and L_1 by* (3.56)*; then $\widetilde{A}$ represents the boundary condition* (3.58)*, more explicitly written as* (3.59).

(ii) *For any $\lambda \in \varrho(A_\gamma)$, $\widetilde{A}-\lambda$ corresponds to the operator $L_1^\lambda : X_1 \to Y_1$ defined in* (3.61)*, and represents the boundary condition* (3.62)*, also written as in* (3.63), (3.64)*. Here L_1^λ and L_1 are related by* (3.65)*, and*

$$\begin{aligned} \ker(\widetilde{A}-\lambda) &= K_\gamma^\lambda \Lambda_{\frac12} \ker L_1^\lambda, \\ \operatorname{ran}(\widetilde{A}-\lambda) &= \Gamma_{0,W_{\bar\lambda}}^* \operatorname{ran} L_1^\lambda + (H \ominus W_{\bar\lambda}). \end{aligned} \tag{3.66}$$

P r o o f. In view of the preparations before the theorem, it remains to account for (3.66), which follows by application of the various transformation maps to (2.43). □

The M-function in this set-up is defined from formula (2.41), in a similar way as in (3.41):

$$\begin{aligned} M_{L_1}(\lambda) &= \Gamma_{0,V} M_{\widetilde{A}}(\lambda) \Gamma_{0,W}^* = \Lambda_{-\frac12}\gamma_V M_{\widetilde{A}}(\lambda)\Gamma_{0,W}^* \\ &= \Lambda_{-\frac12}\gamma_V \operatorname{pr}_\zeta \big((I - (\widetilde{A}-\lambda)^{-1}(A_{\max}-\lambda)\big) A_\gamma^{-1} \operatorname{i}_{W\to H} \Gamma_{0,W}^* \\ &= \Lambda_{-\frac12}\gamma_0 \big(I - (\widetilde{A}-\lambda)^{-1}(A_{\max}-\lambda)\big) A_\gamma^{-1} \operatorname{i}_{W\to H} \Gamma_{0,W}^*. \end{aligned} \tag{3.67}$$

Here, when $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$, $-M_{L_1}(\lambda)$ is the inverse of L_1^λ. We therefore have the following:

Theorem 3.10 *Once again, let A be the second-order strongly elliptic operator introduced above and let the closed realisation $\widetilde{A}$ correspond to $T : V \to W$ by Theorem* 2.1.

The M-function in this setting is

$$M_{L_1}(\lambda) = \Lambda_{-\frac{1}{2}}\gamma_0\big(I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda)\big)A_\gamma^{-1}\,\mathrm{i}_{W\to H}\,\Gamma_{0,W}^*, \tag{3.68}$$

a family of bounded operators from Y_1 to X_1, depending holomorphically on $\lambda \in \varrho(\widetilde{A})$. For $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$,it satisfies

$$M_{L_1}(\lambda) = -(L_1 + \mathrm{pr}_{Y_1}\Lambda_{\frac{1}{2}}(P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1})\Lambda_{\frac{1}{2}}\,\mathrm{i}_{X_1\to L_2})^{-1}. \tag{3.69}$$

The following resolvent formula holds for all $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$:

$$\begin{aligned}(\widetilde{A} - \lambda)^{-1} &= (A_\gamma - \lambda)^{-1} - \mathrm{i}_{V_\lambda\to H}\,\Gamma_{0,V_\lambda}^{-1}M_{L_1}(\lambda)(\Gamma^*_{0,W_{\bar\lambda}})^{-1}\,\mathrm{pr}_{W_{\bar\lambda}}\\ &= (A_\gamma - \lambda)^{-1} - K^\lambda_{\gamma,X_1}M_{L_1}(\lambda)(K'^{\bar\lambda}_{\gamma,Y_1})^*,\end{aligned} \tag{3.70}$$

where $K^\lambda_{\gamma,X_1} : X_1 \to H$ acts like the composition of $\Lambda_{\frac{1}{2}} : X_1 \to X$, $\mathrm{i}_{X\to H^{-\frac{1}{2}}}$ and $K^\lambda_\gamma : H^{-\frac{1}{2}} \to H$.

P r o o f. It remains to show (3.70), which follows from (2.42). □

The analysis of the realisations corresponding to operators $T : Z \to Z'$ covers all the most frequently studied boundary conditions for second-order scalar elliptic operators, whereas the cases where T acts between nontrivial subspaces of Z and Z' are more exotic. For example, the Zaremba problem, where Dirichlet resp. Neumann conditions are imposed on two closed subsets Σ_D resp. Σ_N of Σ with common boundary and covering Σ, leads to the subspace $V = K^0_\gamma H_0^{-\frac{1}{2}}(\Sigma_N)$, which presents additional technical difficulties not covered by the ψdbo calculus.

But when we go beyond the scalar second-order case, subspace situations have a primary interest; see the next section.

3.3 Higher order operators and systems

Let us now consider systems (matrix-formed operators) and higher order elliptic operators. Here one finds that subspace cases occur very naturally and allow studies within the ψdbo calculus, with much the same flavour as in Theorems 3.3, 3.4 and 3.6. For even-order operators, a general and useful framework was worked out in [17] — normal boundary conditions for operators acting between vector bundles — which could be the point of departure for ample generalisations. Scalar operators are covered by a simpler analysis in [16]. We shall here show in detail how the analysis of [16] can be used, and illustrate systems cases by examples.

Example 3.11 Let $A = (A_{jk})$ be an $p \times p$-matrix of second-order differential operators on Ω, elliptic in the sense that the determinant of the principal symbol is nonzero for $x \in \overline{\Omega}$, $\xi \neq 0$. We here have a Green's formula like (3.7), but where $s_0(x)$ is a regular $p \times p$-matrix and $\bar{s}_0(x)$ is replaced by $s_0(x)^*$. There is again a Dirichlet realisation A_γ, and if it is elliptic and $0 \in \varrho(A_\gamma)$, we can repeat the study of Section 3.2, now for p-vectors. That will cover boundary conditions of the form (3.28), and give a somewhat abstract treatment of the more general cases.

Now one can also consider boundary conditions where a Dirichlet condition is imposed on some components of $\gamma_0 u$ and a Neumann condition is imposed on other components of $\nu_1 u$. This is very simple to explain when $s_0(x) = I$, so let us consider that case. Let $\widetilde{A}$ be determined by a boundary condition of the type

$$\gamma_0 u_{N_0} = 0, \quad \gamma_1 u_{N_1} - C\gamma_0 u_{N_1} = 0, \tag{3.71}$$

where N_0 and N_1 are complementing subsets of the index set $N = \{1, 2, \dots, p\}$ with p_0 resp. p_1 elements. Using the homeomorphism $\gamma_Z : Z \xrightarrow{\sim} \prod_{1\le j\le p} H^{-\frac{1}{2}}(\Sigma)$ and similar vector valued versions of the other mappings in Section 3.2, one finds that realisations of boundary conditions (3.71) correspond to operators

$$L : \prod_{j\in N_1} H^{-\frac{1}{2}}(\Sigma) \to \prod_{j\in N_1} H^{\frac{1}{2}}(\Sigma), \quad L = C - \mathrm{pr}_{N_1}\,P^0_{\gamma_0,\gamma_1 i}\,\mathrm{i}_{N_1}, \tag{3.72}$$

where pr_{N_1} projects $\{\varphi_j\}_{j\in N}$ onto $\{\varphi_j\}_{j\in N_1}$, and i_{N_1} injects vectors $\{\varphi_j\}_{j\in N_1}$ into vectors indexed by N by supplying with zeroes at the places indexed by N_0. Again, ellipticity makes the realisation regular, and there are formulas very similar to those in Section 3.2:

$$L^\lambda = L + \mathrm{pr}_{N_1}(P^0_{\gamma_0,\gamma_1} - P^\lambda_{\gamma_0,\gamma_1})\,\mathrm{i}_{N_1} = C - \mathrm{pr}_{N_1} P^\lambda_{\gamma_0,\gamma_1}\,\mathrm{i}_{N_1}, \quad M_L(\lambda) = -(L^\lambda)^{-1}, \tag{3.73}$$
$$(\widetilde{A}-\lambda)^{-1} = (A_\gamma - \lambda)^{-1} - K^\lambda_\gamma\,\mathrm{i}_{N_1}\, M_L(\lambda)\,\mathrm{pr}_{N_1}(K'^{\bar\lambda}_\gamma)^*,$$

where L^λ and $M_L(\lambda)$ are elliptic ψdo's when the boundary condition is elliptic.

Notation gets a little more complicated if s_0 is not in diagonal form, or if pr_{N_1} is replaced by a projection onto a p_1-dimensional subspace of $\mathbb{C}^p$ that varies with $x\in\Sigma$. Then it is useful to apply vector bundle notation, as in [17], where fully general boundary conditions are treated.

Next, consider the case where A is a $2m$-order operator $A = \sum_{|\alpha|\le 2m} a_\alpha(x)D^\alpha$ (scalar or matrix-formed), elliptic on $\overline{\Omega}$. The Cauchy data are the boundary values

$$\varrho u = \{\gamma_0 u, \dots, \gamma_{2m-1}u\}, \tag{3.74}$$

which one can split into the Dirichlet data and the Neumann data

$$\gamma u = \{\gamma_0 u,\dots,\gamma_{m-1}u\}, \quad \nu u = \{\gamma_m u,\dots,\gamma_{2m-1}u\}. \tag{3.75}$$

There is a Green's formula for $u,v\in H^{2m}(\Omega)$:

$$(Au,v) - (u,A'v) = (\chi u,\gamma v) - (\gamma u,\chi' v), \quad \chi u = \mathcal{A}_1\nu u, \quad \chi' v = \mathcal{A}'_1\nu v + \mathcal{A}'_0\gamma v, \tag{3.76}$$

where $\mathcal{A}_1$ and $\mathcal{A}'_1$ (both invertible) and $\mathcal{A}'_0$ are suitable $m\times m$ matrices of differential operators over Σ. Here $\chi = \{\chi_0,\dots,\chi_{m-1}\}$ with χ_j of order $2m-j-1$, and the trace maps are continuous,

$$\gamma : Z^s_\lambda(A) \to \prod_{0\le j\le m-1} H^{s-j-\frac12}(\Sigma), \quad \chi : Z^s_\lambda(A) \to \prod_{0\le j\le m-1} H^{s-2m+j+\frac12}(\Sigma), \tag{3.77}$$

for $s\in\mathbb{R}$. If the Dirichlet problem is uniquely solvable, one can again use A_γ as reference operator, and set

$$P^\lambda_{\gamma,\chi} = \chi K^\lambda_\gamma; \quad P'^{\bar\lambda}_{\gamma,\chi'} = \chi' K'^{\bar\lambda}_\gamma; \tag{3.78}$$

with K^λ_γ resp. $K'^{\bar\lambda}_\gamma$ denoting the Poisson operator solving

$$(A-\lambda)u = 0 \text{ resp. } (A'-\bar\lambda)u = 0 \text{ in } \Omega, \quad \gamma u = \varphi. \tag{3.79}$$

Now

$$\Gamma = \chi - P^0_{\gamma,\chi}\gamma \ \text{ and } \ \Gamma' = \chi' - P'^0_{\gamma,\chi'}\gamma \tag{3.80}$$

are defined as continuous maps from $D(A_{\max})$, resp. $D(A'_{\max})$ to $\prod_{j<m} H^{j+\frac12}(\Sigma)$, and there is a generalisation of (3.76) valid for all $u\in D(A_{\max})$, $v\in D(A'_{\max})$:

$$(Au,v) - (u,A'v) = (\Gamma u,\gamma v)_{\{j+\frac12\},\{-j-\frac12\}} - (\gamma u,\Gamma' v)_{\{-j-\frac12\},\{j+\frac12\}} \tag{3.81}$$

(where $(\cdot,\cdot)_{\{s_j\},\{-s_j\}}$ indicates the duality pairing between $\prod_{j<m} H^{s_j}(\Sigma)$ and $\prod_{j<m} H^{-s_j}(\Sigma)$).

It should be noted that $P^\lambda_{\gamma,\chi}$ is a *mixed-order system*:

$$P^\lambda_{\gamma,\chi} = (P_{jk})_{j,k=0,\dots,m-1}; \quad P_{jk} \text{ of order } 2m-1-j-k,$$

and the principal symbol and possible ellipticity is defined accordingly. Such systems (with differential operator entries) were first considered by Douglis and Nirenberg [11] and by Volevich [44]; in the elliptic case they are called Douglis-Nirenberg elliptic.

Formula (3.81) can be turned into a boundary triplet formula with $\mathcal{H} = \mathcal{K} = L_2(\Sigma)^m$ by composition of Γ and Γ' with a symmetric ψdo defining the norms:

$$\Theta = (\delta_{kl}\Lambda_{k+\frac{1}{2}})_{k,l=0,\dots,m-1} : \prod_{0\le j<m} H^{j+\frac{1}{2}}(\Sigma) \xrightarrow{\sim} L_2(\Sigma)^m, \tag{3.82}$$

and composition of γ with Θ^{-1}, setting

$$\Gamma_1 = \Theta\Gamma, \quad \Gamma_1' = \Theta\Gamma', \quad \Gamma_0 = \Theta^{-1}\gamma = \Gamma_0'; \tag{3.83}$$

this leads to a word-for-word generalisation of Proposition 3.1, with

$$(Au, v)_{L_2(\Omega)} - (u, A'v)_{L_2(\Omega)} = (\Gamma_1 u, \Gamma_0' v)_{L_2(\Sigma)^m} - (\Gamma_0 u, \Gamma_1' v)_{L_2(\Sigma)^m}, \tag{3.84}$$

for $u \in D(A_{\max})$, $v \in D(A'_{\max})$.

The straightforward continuation of what we did in Section 3.2 is the study of boundary conditions of the type $\chi u = C\gamma u$, that are the ones obtained when $\widetilde{A}$ corresponds to $T : Z \to Z'$. When $\widetilde{A}$ corresponds to $T : V \to W$, we find that it represents a boundary condition similar to (3.58). Since the proofs in Section 3.2 generalise immediately to this situation, we can state:

Corollary 3.12 *Theorems* 3.3, 3.4 *and* 3.6 *extend to the present situation for elliptic operators of order* $2m$, *when* $H^{-\frac{1}{2}}(\Sigma)$ *is replaced by* $\prod_{j<m} H^{-j-\frac{1}{2}}(\Sigma)$, *and* γ_0 *and* ν_1 *are replaced by* γ *and* χ.

Theorems 3.9 *and* 3.10 *likewise extends when, moreover,* $L_2(\Sigma)$ *is replaced by* $L_2(\Sigma)^m$ *and* (3.84) *is used.*

Whereas the statements in the case of general subspaces V and W will be somewhat abstract, there are now also formulations where the subspaces are represented by products of Sobolev spaces, and the operators belong to the ψdbo calculus. We shall demonstrate this on the basis of the treatment of normal boundary value problems in [16], which we now recall. Denote

$$M = \{0, \dots, 2m-1\}, \quad M_0 = \{0, \dots, m-1\}, \quad M_1 = \{m, \dots, 2m-1\}.$$

A general normal boundary condition is given as

$$\gamma_j u + \sum_{k<j} B_{jk}\gamma_k u = 0, \quad j \in J, \tag{3.85}$$

where J is a subset of M with m elements and the B_{jk} are differential operators on Σ of order $j-k$. (It is called normal since the highest-order trace operators γ_j have coefficient 1.) Let $K = M \setminus J$, and set

$$\begin{aligned} &J_0 = J \cap M_0, \quad J_1 = J \cap M_1, \quad K_0 = K \cap M_0, \quad K_1 = K \cap M_1, \\ &\gamma_{J_0} = \{\gamma_j\}_{j\in J_0}, \quad \nu_{J_1} = \{\gamma_j\}_{j\in J_1}, \quad \gamma_{K_0} = \{\gamma_j\}_{j\in K_0}, \quad \nu_{K_1} = \{\gamma_j\}_{j\in K_1}, \end{aligned}$$

then (3.85) can be reduced to the form

$$\gamma_{J_0} u = F_0\gamma_{K_0} u, \quad \nu_{J_1} u = F_1\gamma_{K_0} u + F_2\nu_{K_1} u, \tag{3.86}$$

with suitable matrices of differential operators F_0, F_1, F_2. To reformulate this in terms of $\{\gamma, \chi\}$, we use the convention for reflected sets:

$$N' = \{j \mid 2m-1-j \in N\}, \tag{3.87}$$

considered as an ordered subset of M. Then $\chi = \{\chi_j\}_{j\in M_0}$ splits into $\chi = \{\chi_{J_1'}, \chi_{K_1'}\}$, where

$$\chi_{J_1'} = \{\chi_j\}_{j\in J_1'}, \quad \chi_{K_1'} = \{\chi_j\}_{j\in K_1'}.$$

We can now reformulate (3.86) as

$$\gamma_{J_0} u = F_0\gamma_{K_0} u, \quad \chi_{J_1'} u = G_1\gamma_{K_0} u + G_2\chi_{K_1'} u, \tag{3.88}$$

where G_1 and G_2 are matrices of differential operators derived from those in (3.86) and the coefficients in Green's formula (more details in [16]).

When $\widetilde{A}$ is the realisation of A determined by the boundary condition (3.88) (i.e., $D(\widetilde{A})$ consists of the $u \in D(A_{\max})$ satisfying (3.88)), and ellipticity holds, then $D(\widetilde{A}) \subset H^{2m}(\Omega)$, and the adjoint realisation $\widetilde{A}^*$ is determined by the likewise elliptic boundary condition

$$\gamma_{K_1{}'}v = -G_2^*\gamma_{J_1{}'}v, \quad \chi'_{K_0}v = G_1^*\gamma_{J_1{}'}v - F_0^*\chi'_{J_0}v. \tag{3.89}$$

Setting

$$\Phi = \begin{pmatrix} I_{K_0K_0} \\ F_0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} I_{J_1{}'J_1{}'} \\ -G_2^* \end{pmatrix},$$

we have that

$$\gamma D(\widetilde{A}) \subset X = \Phi\big(\prod_{k\in K_0} H^{-k-\frac12}(\Sigma)\big), \quad \gamma D(\widetilde{A}^*) \subset Y = \Psi\big(\prod_{j\in J_1{}'} H^{-j-\frac12}(\Sigma)\big).$$

Here X is the graph of F_0 and naturally homeomorphic to its "first component" $\prod_{k\in K_0} H^{-k-\frac12}(\Sigma)$, and similarly Y is homeomorphic to $\prod_{j\in J_1{}'} H^{-j-\frac12}(\Sigma)$. The operator $T : V \to W$ that $\widetilde{A}$ corresponds to by Theorem 2.1 carries over to an operator $L : X \to Y^*$ by use of γ, and this is further reduced to an operator

$$L_1 : \prod_{k\in K_0} H^{-k-\frac12}(\Sigma) \to \prod_{j\in J_1{}'} H^{j+\frac12}(\Sigma), \quad L_1 = G_1 - \Psi^* P^0_{\gamma,\chi}\Phi,$$

with domain $D(L_1) = \prod_{k\in K_0} H^{2m-k-\frac12}(\Sigma)$. This representation is used in [16] to find criteria for the operator to be m-accretive, and the ideas are further pursued in [17] for systems of operators (where vector bundle notation is needed).

For the present study of resolvents, we now find that $\widetilde{A} - \lambda$ can be represented by

$$L_1^\lambda = G_1 - \Psi^* P^\lambda_{\gamma,\chi}\Phi : \prod_{k\in K_0} H^{-k-\frac12}(\Sigma) \to \prod_{j\in J_1{}'} H^{j+\frac12}(\Sigma), \quad D(L_1^\lambda) = \prod_{k\in K_0} H^{2m-k-\frac12}(\Sigma), \tag{3.90}$$

when $\lambda \in \varrho(A_\gamma)$. The corresponding M-function and Kreĭn formula are:

$$\begin{aligned} M(\lambda) &= -(G_1 - \Psi^* P^\lambda_{\gamma,\chi}\Phi)^{-1} : \prod_{j\in J_1{}'} H^{j+\frac12}(\Sigma) \to \prod_{k\in K_0} H^{-k-\frac12}(\Sigma), \\ (\widetilde{A} - \lambda)^{-1} &= (A_\gamma - \lambda)^{-1} - K^\lambda_\gamma \Phi M(\lambda)(K'^{\bar\lambda}_\gamma \Psi)^*. \end{aligned} \tag{3.91}$$

When $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$, $M(\lambda)$ extends holomorphically to $\varrho(\widetilde{A})$ (note that the spectrum of $\varrho(A_\gamma)$ is discrete in this case). As a mixed-order operator with entries of order $2m-1-j-k$ ($k \in K_0$, $j \in J_1{}'$), L_1^λ is Douglis-Nirenberg elliptic, and $M(\lambda)$ is so in the opposite direction.

The two functions $M(\lambda)$ and L_1^λ together give a tool to analyse the spectral properties of $\widetilde{A}$ in terms of ψdo's on Σ, $M(\lambda)$ being holomorphic on $\varrho(\widetilde{A})$ and L_1^λ containing information on null-spaces and ranges. We have hereby obtained:

Theorem 3.13 *Let A be a $2m$-order elliptic differential operator with coefficients in $C^\infty(\overline{\Omega})$, and let $\widetilde{A}$ be the realisation defined by the normal boundary condition* (3.85)*, reformulated as* (3.88)*; assume that the boundary problem is elliptic. Let A_γ be the Dirichlet realisation, assumed elliptic and invertible. For $\lambda \in \varrho(A_\gamma)$, the operator corresponding to $\widetilde{A} - \lambda$ by Corollary* 2.3 *carries over to* (3.90)*. The associated M-function and Kreĭn resolvent formula are described in* (3.91)*; $M(\lambda)$ extends to an operator family holomorphic in $\lambda \in \varrho(\widetilde{A})$.*

When A is scalar of order $2m$, this analysis covers *all* the elliptic problems to which Seeley's resolvent construction [40] applies, for it is known that normality of the boundary condition is necessary for the parameter-ellipticity required there (cf. e.g. [21, Section 1.5]). When $p \times p$-systems are considered, each line in (3.85) can moreover be composed with a multiplication map to the sections of a subbundle of $\Sigma \times \mathbb{C}^p$, as indicated in Example 3.11; here normality means surjectiveness of the coefficient of the highest normal derivative γ_j, for each j. For such cases, the treatment can be based on [17].

Example 3.14 For a simple example, consider the biharmonic operator $A = \Delta^2$, which has the Green's formula

$$(\Delta^2 u, v) - (u, \Delta^2 v) = (\chi u, \gamma v) - (\gamma u, \chi v), \quad \chi u = \begin{pmatrix} -\gamma_1 \Delta u \\ \gamma_0 \Delta u \end{pmatrix}; \tag{3.92}$$

note that linear conditions on $\gamma_0 u$, $\gamma_1 u$, $\gamma_2 u$ and $\gamma_3 u$ can be written as conditions on $\gamma_0 u$, $\gamma_1 u$, $\gamma_0 \Delta u$ and $\gamma_1 \Delta u$, hence on γu and χu. The operator $P^\lambda_{\gamma,\chi}$ is a ψdo from $H^{s-\frac12} \times H^{s-\frac32}$ to $H^{s-\frac72} \times H^{s-\frac52}$ (note the reverse order of the Sobolev exponents in the target space).

Let $\widetilde{A}$ be defined by an elliptic boundary condition with a first-order differential operator C,

$$\gamma_0 u = 0, \quad \gamma_0 \Delta u = C \gamma_1 u; \tag{3.93}$$

then $\widetilde{A}^*$ represents the likewise elliptic boundary condition

$$\gamma_0 u = 0, \quad \gamma_0 \Delta u = C^* \gamma_1 u, \tag{3.94}$$

and the domains of $\widetilde{A}$ and $\widetilde{A}^*$ are contained in $H^4(\Omega)$. For the corresponding operator $T : V \to W$ according to Theorem 2.1, $V = W = K^0_\gamma(\{0\} \times H^{-\frac32})$. This carries over to the operator

$$L = C - \mathrm{pr}_2 \, P^0_{\gamma,\chi} \, \mathrm{i}_2 : H^{-\frac32} \to H^{\frac32}, \tag{3.95}$$

where $\mathrm{pr}_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$ and $\mathrm{i}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thanks to the ellipticity, $D(L) = H^{\frac52}$, and L is an elliptic ψdo of order 1. We here find that

$$\begin{aligned} &L^\lambda = L + \mathrm{pr}_2 (P^0_{\gamma,\chi} - P^\lambda_{\gamma,\chi}) \, \mathrm{i}_2 = C - \mathrm{pr}_2 \, P^\lambda_{\gamma,\chi} \, \mathrm{i}_2, \quad M_L(\lambda) = -(L^\lambda)^{-1}, \\ &(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} - K^\lambda_\gamma \, \mathrm{i}_2 \, M_L(\lambda) \, \mathrm{pr}_2 (K'^{\bar\lambda}_\gamma)^*. \end{aligned} \tag{3.96}$$

It is also possible to take another realisation A_β than the Dirichlet realisation as reference operator, preferably one defined by an elliptic boundary condition, as in [15]. For strongly elliptic operators, using A_γ as reference operator has the advantage that semiboundedness properties are preserved in correspondences between $\widetilde{A}$ and T, see [16, 17].

There do exist even-order operators for which the Dirichlet problem is not elliptic; a well-known example is the operator $-\Delta I + 2 \,\mathrm{grad}\,\mathrm{div}$ on subsets of $\mathbb{R}^2$, studied by Bitsadze.

Example 3.15 The operator A need not be of even order. For example, first-order $p \times p$-systems, such as Dirac operators, have received much attention. In this case the Cauchy data are $\{\gamma_0 u_1, \dots, \gamma_0 u_p\}$. Elliptic boundary conditions require p to be even. In "lucky" cases, one can get an elliptic boundary value problem by imposing the vanishing of half of the boundary values; this will then give a reference problem, allowing the discussion of other realisations. More systematically, one can impose the condition $\Pi_+ \gamma_0 u = 0$ for a certain ψdo projection Π_+ over the boundary (the Atiyah-Patodi-Singer condition), which defines a Fredholm realisation. If it is invertible, it can be used as a reference operator.

Similar considerations can be worked out for elliptic operators on suitable unbounded domains, and on manifolds. More specifically, the calculus of ψdbo's is extended in [21] to the manifolds called "admissible" there; they have finitely many unbounded ends with good control over coordinate diffeomorphisms. They include complements in $\mathbb{R}^n$ as well as $\overline{\mathbb{R}}^n_+$ of smooth bounded domains.

Example 3.16 We shall here make concrete the considerations in Example 3.14 for the biharmonic operator on the half-space

$$\mathbb{R}^n_+ = \{x = \{x', x_n\} \in \mathbb{R}^n \mid x_n > 0\},$$

where we denote $\{x_1, \dots, x_{n-1}\} = x'$. This constant-coefficient case can be viewed as a model for variable-coefficient cases, giving an example of the pointwise symbol calculations entering in the ψdbo theory. (A detailed introduction to the ψdbo calculus is found e.g. in [22].)

It is well-known that since Δ^2 is symmetric nonnegative, the resolvent of the Dirichlet realisation in $L_2(\mathbb{R}^n_+)$ exists for $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$. We shall write

$$\lambda = -\mu^4, \quad \mu \in V_{\pi/4}, \text{ where } V_\theta = \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \theta\}.$$

We fix a $\lambda_0 < 0$, then the realisations of $A = \Delta^2 - \lambda_0$ fit into the general set-up, with the Dirichlet realisation A_γ of $\Delta^2 - \lambda_0$ as reference operator. However, for simplicity in formulas (avoiding addition and subtraction of λ_0), we keep the parameter λ for the operator families defined relative to $\Delta^2 - \lambda$.

In the following calculations, one can think of a $\mu > 0$; the considerations extend holomorphically to $V_{\pi/4}$. To find the Poisson operator K^λ_γ solving the problem

$$(\Delta^2 - \lambda)u(x', x_n) = 0 \text{ on } \mathbb{R}^n_+, \quad u(x', 0) = \varphi_0(x'), \quad \partial_n u(x', 0) = \varphi_1(x'),$$

we perform a Fourier transformation in the x'-variable, and then have to find solutions of

$$\begin{aligned} ((|\xi'|^2 - \partial_n^2)^2 + \mu^4)\acute{u}(\xi', x_n) &= 0 \text{ for } x_n > 0, \\ \acute{u}(\xi', 0) &= \hat{\varphi}_0(\xi'), \\ \partial_n \acute{u}(\xi', 0) &= \hat{\varphi}_1(\xi'), \end{aligned} \tag{3.97}$$

that are in $\mathcal{S}(\overline{\mathbb{R}}_+) = r^+\mathcal{S}(\mathbb{R})$ (the restriction to $\mathbb{R}_+$ of the Schwartz space of rapidly decreasing functions on $\mathbb{R}$). Write

$$\begin{aligned} ((|\xi'|^2 - \partial_n^2)^2 + \mu^4) &= (|\xi'|^2 + i\mu^2 - \partial_n^2)(|\xi'|^2 - i\mu^2 - \partial_n^2) \\ &= (\sigma_+ + \partial_n)(\sigma_+ - \partial_n)(\sigma_- + \partial_n)(\sigma_- - \partial_n), \\ \sigma_+ = (|\xi'|^2 + i\mu^2)^{\frac12}, &\quad \sigma_- = (|\xi'|^2 - i\mu^2)^{\frac12}, \end{aligned} \tag{3.98}$$

where $z^{\frac12}$ is defined for $z \in V_\pi$ to be real positive for $z \in \mathbb{R}_+$; note that $\mu^2 \in V_{\pi/2}$ so that $|\xi'|^2 \pm i\mu^2 \in V_\pi$, and $\operatorname{Re}\sigma_\pm > 0$. Then the general solution in $\mathcal{S}(\overline{\mathbb{R}}_+)$ of the first line in (3.97) is

$$v(x_n) = c_1 e^{-\sigma_+ x_n} + c_2 e^{-\sigma_- x_n}.$$

It is adapted to the boundary conditions by solution of

$$c_1 + c_2 = \hat{\varphi}_0(\xi'), \quad -\sigma_+ c_1 - \sigma_- c_2 = \hat{\varphi}_1(\xi'),$$

with respect to (c_1, c_2); this gives

$$\acute{u}(\xi', x_n) = \tfrac{1}{\sigma_+ - \sigma_-}\begin{pmatrix} e^{-\sigma_+ x_n} & e^{-\sigma_- x_n} \end{pmatrix} \begin{pmatrix} -\sigma_- & -1 \\ \sigma_+ & 1 \end{pmatrix} \begin{pmatrix} \hat{\varphi}_0 \\ \hat{\varphi}_1 \end{pmatrix} \equiv \tilde{k}^\lambda_\gamma \begin{pmatrix} \hat{\varphi}_0 \\ \hat{\varphi}_1 \end{pmatrix}.$$

Here $\tilde{k}^\lambda_\gamma(\xi', x_n)$ is the so-called symbol-kernel of the Poisson operator K^λ_γ; it acts like

$$K^\lambda_\gamma \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \tilde{k}^\lambda_\gamma(\xi', x_n) \begin{pmatrix} \hat{\varphi}_0(\xi') \\ \hat{\varphi}_1(\xi') \end{pmatrix} d\xi',$$

and maps $H^{s-\frac12}(\mathbb{R}^{n-1}) \times H^{s-\frac32}(\mathbb{R}^{n-1})$ to $\{u \in H^s(\mathbb{R}^n_+) \mid (A-\lambda)u = 0\}$ for all $s \in \mathbb{R}$. (In variable-coefficient cases, $\tilde{k}$ would moreover depend on x'.)

The symbol $p^\lambda(\xi')$ of the Dirichlet-to-Neumann operator $P^\lambda_{\gamma,\chi}$ with χ chosen as in (3.92) is found by the calculation

$$\begin{aligned} p^\lambda(\xi') &= \gamma_0 \begin{pmatrix} \partial_n(|\xi'|^2 - \partial_n^2) \\ -|\xi'|^2 + \partial_n^2 \end{pmatrix} \tilde{k}^\lambda_\gamma(\xi', x_n) = \frac{1}{\sigma_+ - \sigma_-} \begin{pmatrix} -i\sigma_+\mu^2 & i\sigma_-\mu^2 \\ i\mu^2 & -i\mu^2 \end{pmatrix} \begin{pmatrix} -\sigma_- & -1 \\ \sigma_+ & 1 \end{pmatrix} \\ &= \frac{i\mu^2}{\sigma_+ - \sigma_-} \begin{pmatrix} 2\sigma_+\sigma_- & \sigma_+ + \sigma_- \\ -(\sigma_+ + \sigma_-) & -2 \end{pmatrix} \\ &= \tfrac14(\sigma_+ + \sigma_-) \begin{pmatrix} 2\sigma_+\sigma_- & \sigma_+ + \sigma_- \\ -(\sigma_+ + \sigma_-) & -2 \end{pmatrix} = \begin{pmatrix} p^\lambda_{00} & p^\lambda_{01} \\ p^\lambda_{10} & p^\lambda_{11} \end{pmatrix}; \end{aligned} \tag{3.99}$$

here we used that $(\sigma_+ - \sigma_-)^{-1} = (\sigma_+ + \sigma_-)/(\sigma_+^2 - \sigma_-^2) = (\sigma_+ + \sigma_-)/(4i\mu^2)$.

The operator $P^{\lambda}_{\gamma,\chi}$ is $\operatorname{Op}(p^\lambda)$, where we use the notation for ψdo's

$$\operatorname{Op}(q)f = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} q(x',\xi')\hat{f}(\xi')\, d\xi'.$$

Let us consider a model boundary condition (3.93) for $\Delta^2 - \lambda_0$ with a first-order differential operator $C = b_1\partial_1 + \cdots + b_{n-1}\partial_{n-1}$ with constant coefficients; it defines the realisation $\widetilde{A}$, and C has symbol $c(\xi') = ib\cdot\xi'$.

Then the operators L^{λ_0} and L^λ defined similarly to (3.95) and (3.96) are the ψdo's with symbol

$$\begin{aligned} l^{\lambda_0}(\xi') &= c(\xi') - p^{\lambda_0}_{11}(\xi'), \text{ resp.} \\ l^{\lambda}(\xi') &= l^{\lambda_0}(\xi') + p^{\lambda_0}_{11}(\xi') - p^{\lambda}_{11}(\xi') = c(\xi') - p^{\lambda}_{11}(\xi') = c(\xi') + \tfrac{1}{2}(\sigma_+ + \sigma_-). \end{aligned}$$

When l^λ is invertible for all ξ', $M_L(\lambda)$ is the ψdo with symbol

$$m(\xi',\lambda) = -\big(c(\xi') + \tfrac{1}{2}(\sigma_+ + \sigma_-)\big)^{-1} = -\big(c(\xi') + \tfrac{1}{2}(|\xi'|^2 + i\mu^2)^{\frac{1}{2}} + \tfrac{1}{2}(|\xi'|^2 - i\mu^2)^{\frac{1}{2}}\big)^{-1}. \tag{3.100}$$

This shows how $M_L(\lambda)$ is found. As an additional observation, we remark that when b is real nonzero, then $C^* = -C \neq 0$, so $\widetilde{A}$ is non-selfadjoint, cf. (3.94). However, $M_L(\lambda)$ is well-defined for all $\lambda \in \mathbb{C}\setminus\overline{\mathbb{R}}_+$ (since $c(\xi') \in i\mathbb{R}$). On the other hand, L^{λ_0} has numerical range $\nu(L^{\lambda_0})$ approximately equal to the sector $V = \{z \in \mathbb{C} \mid |\operatorname{Im} z| \le |b| \operatorname{Re} z\}$, and $\nu(L^{\lambda_0})$ is contained in the numerical range of $\widetilde{A}$, by [15, Th. III 4.3]. This gives an example where $\widetilde{A}$ has a large numerical range outside the spectrum.

Remark 3.17 One can ask whether the considerations extend to Douglis-Nirenberg elliptic systems (systems of mixed order). But it may not be easy. Consider for example a 2×2-system

$$Aw = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{3.101}$$

on a smooth bounded open set $\Omega \subset \mathbb{R}^n$, where u and v are scalar, and A_{ij} is of order $4-i-j$. There is a Green's formula for A:

$$(Aw, w') - (w, A^*w') = (\kappa w, \gamma_0 u') - (\gamma_0 u, \kappa' w'),$$

where $\kappa w = \chi u + b_1\gamma_0 v$, $\kappa' w' = \chi' u' + b_2\gamma_0 v'$, with first-order trace operators χ and χ'; $\{\gamma_0 u, \kappa w\}$ are the *reduced Cauchy data* according to [13, 18]. Assume that the principal symbol is uniformly positive definite; in particular, the function $A_{22}(x) \ge c > 0$, the symbol $a^0_{11}(x,\xi) \ge c|\xi|^2$ and the determinant $a^0_{11}(x,\xi)A_{22}(x) - a^0_{12}(x,\xi)a^0_{21}(x,\xi) \ge c|\xi|^2$. Then the Dirichlet problem for A (with boundary condition $\gamma_0 u = 0$) is well-posed, with domain $D(A_\gamma) = (H^2(\Omega)\cap H^1_0(\Omega)) \times H^1(\Omega)$.

When $0 \in \varrho(A_\gamma)$, there is the following parametrization of null-spaces of A (cf. also [18, Th. 2.11]):

$$\gamma_0 \operatorname{pr}_1 : Z_0^{s,s-1}(A) = \{w \in H^s(\Omega)\times H^{s-1}(\Omega) | Aw = 0\} \xrightarrow{\sim} H^{s-\frac{1}{2}}(\Sigma),$$

all $s \in \mathbb{R}$; but this does not in general lead to a nice parametrization of

$$Z_0^0(A) = \{w \in L_2(\Omega)^2 | Aw = 0\},$$

which would be required to get a good interpretation of the abstract theory for A as an operator in $L_2(\Omega)^2$.

This regularity problem is not present in the one-dimensional situation, where the maximal domain is $H^2\times H^1$; an example is considered in Section 4.2.

In the example in Section 4.1 we consider an n-dimensional case of (3.101), where the off-diagonal terms are of order 0; this allows an easier parametrization of the null-space.

4 Examples

We have seen that the family $M_{\widetilde{A}}(\lambda)$ is holomorphic on $\varrho(\widetilde{A})$ so that

$$\sigma(\widetilde{A}) \supset \{\lambda \in \mathbb{C} \mid M_{\widetilde{A}}(\lambda) \text{ singular at } \lambda\}. \tag{4.1}$$

Among the singular points, we have very good control of those outside $\varrho(A_\beta)$ by Corollary 2.12 and (2.43), for the null-space and range of $\widetilde{A}-\lambda$ are fully clarified by the same concepts for T^λ (a holomorphic operator family on $\varrho(A_\beta)$ which is homeomorphic to the inverse of $M_{\widetilde{A}}(\lambda)$ on $\varrho(\widetilde{A}) \cap \varrho(A_\beta)$); it gives the information:

$$\sigma(\widetilde{A}) \cap \varrho(A_\beta) = \{\lambda \in \varrho(A_\beta) \mid \ker T^\lambda \neq \{0\} \text{ or } \operatorname{ran} T^\lambda \neq W_{\bar{\lambda}}\}. \tag{4.2}$$

So the only spectral points of $\widetilde{A}$ whose spectral nature may not be controlled by $M_{\widetilde{A}}(\lambda)$ and T^λ are those that lie in $\sigma(A_\beta)$. For many scalar equations it has long been known that the M-function allows full control of the spectrum. However, when considering systems, it is easy to see that uncontrolled points may exist by considering equations which are decoupled. In [7], an example involving a coupled system of ODEs was given where $M_{\widetilde{A}}$ was regular at a point λ_0 belonging to the essential spectrum of $\widetilde{A}$ (and of A_β). We shall here show a similar phenomenon for PDEs and for a system of ODEs with first order off-diagonal entries.

4.1 PDE counterexample

Consider the 2×2 matrix-formed operator

$$A = \begin{pmatrix} A_0 & a(x) \\ b(x) & c(x) \end{pmatrix}, \tag{4.3}$$

acting on 2-vector functions $w = \{u, v\}$ on Ω, such that A_0 is a second-order elliptic operator as studied in Section 3.2 and $a, b, c \in C^\infty(\overline{\Omega})$. The set $\operatorname{ran} c$ is a compact subset of $\mathbb{C}$. We assume that it has a connected component K with more than one point, that $\mathbb{C} \setminus (\operatorname{ran} c)$ is connected, and that

$$a(x)b(x) \text{ vanishes on } \operatorname{supp} c. \tag{4.4}$$

The maximal and minimal operators are

$$A_{\max} = \begin{pmatrix} A_{0,\max} & a(x) \\ b(x) & c(x) \end{pmatrix}, \quad A_{\min} = \begin{pmatrix} A_{0,\min} & a(x) \\ b(x) & c(x) \end{pmatrix}, \tag{4.5}$$

with

$$D(A_{\max}) = D(A_{0,\max}) \times L_2(\Omega), \quad D(A_{\min}) = D(A_{0,\min}) \times L_2(\Omega), \tag{4.6}$$

and there is the Green's formula

$$\left(A\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix}\right) - \left(\begin{pmatrix} u \\ v \end{pmatrix}, A'\begin{pmatrix} u' \\ v' \end{pmatrix}\right) = (\nu_1 u, \gamma_0 u') - (\gamma_0 u, \nu_1' u' + \mathcal{A}_0' \gamma_0 u')_{L_2(\Sigma)},$$

with notation as in (3.7) ff.

Proposition 4.1 *Let A_γ be the Dirichlet realisation defined by the Dirichlet condition $\gamma_0 u = 0$.*

(i) *A_γ is lower semibounded with domain $(H^2(\Omega) \cap H_0^1(\Omega)) \times L_2(\Omega)$, and the spectrum is contained in a half-space $\{\operatorname{Re}\lambda \geq \alpha\}$.*

(ii) *The operator A_γ has a non-empty essential spectrum, namely*

$$\sigma_{\text{ess}}(A_\gamma) = \operatorname{ran} c. \tag{4.7}$$

(iii) *Outside the essential spectrum, the spectrum is discrete, consisting of eigenvalues of finite multiplicity.*

P r o o f. (i) follows from standard results for elliptic operators and the fact that the adjoint of A_γ is the Dirichlet realisation of A' with similar properties. Hence, the spectrum is contained in a half-space $\{\operatorname{Re}\lambda \geq \alpha\}$.

(ii) follows from Geymonat-Grubb [13], where it was shown that the essential spectrum of the realisation of a mixed-order system A defined by a differential boundary condition $\beta u = 0$ consists exactly of the points $\lambda \in \mathbb{C}$ where ellipticity of $\{A - \lambda, \beta\}$ fails. The current operator A is a mixed-order system with orders 2 and 0 for the diagonal terms, order 1 for the off-diagonal terms (to fit with the rules of Douglis, Nirenberg and Volevich for mixed-order systems). The principal symbol of $A - \lambda$ is

$$\begin{pmatrix} a^0(x,\xi) & 0 \\ 0 & c(x) - \lambda \end{pmatrix}, \tag{4.8}$$

so ellipticity of the Dirichlet problem for $A - \lambda$ fails precisely when $\lambda \in \operatorname{ran} c$.

For (iii), we note that the resolvent set is non-empty, so for $\lambda \notin \operatorname{ran} c$, $A_\gamma - \lambda$ is a Fredholm operator (by the ellipticity) with index 0 (since the index depends continuously on λ). Then all spectral points outside $\operatorname{ran} c$ are eigenvalues with finite dimensional eigenspaces. Since these eigenspaces are linearly independent, there can only be countably many, so there is at most a countable set of eigenvalues outside $\operatorname{ran} c$. They can only accumulate at points of $\operatorname{ran} c$. □

Consider another boundary condition for A,

$$\nu_1 u = C\gamma_0 u, \tag{4.9}$$

with C a first-order differential operator, and such that the system $\{A_0, \nu_1 - C\gamma_0\}$ is elliptic, defining the realisation $\widetilde{A}_0$. Then $\{A, (\nu_1 - C\gamma_0)\operatorname{pr}_1\}$ is likewise elliptic, and we define $\widetilde{A}$ to be the realisation of A under the boundary condition (4.9). Again, the essential spectrum equals $\operatorname{ran} c$.

It is well-known that $\widetilde{A}_0$ satisfies a 1-coerciveness inequality (hence is lower bounded) if and only if the real part of the principal symbol of $L = C - P^0_{\gamma_0,\nu_1}$ is $\geq c_0|\xi'|$ with $c_0 > 0$, and that the adjoint then has similar properties (cf. e.g. [16], [17]). Assuming this, we have that the spectrum lies in a half-plane $\{\operatorname{Re}\lambda \geq \alpha_1\}$, and the spectrum is discrete outside $\operatorname{ran} c$.

We next want to discuss M-functions for the comparison of the Dirichlet realisation and the realisation defined by (4.9). Let k be a point in $\varrho(A_\gamma)$; then the general analysis of Section 2 works for the realisations of $A - k$, with reference operator $A_\gamma - k$. So the holomorphic families T^λ and $M_{(\widetilde{A}-k)}(\lambda)$ are well-defined relative to this set-up. Note that $c(x)$ is now replaced by $c_k(x) = c(x) - k$; the essential spectra of $A_\gamma - k$ and $\widetilde{A} - k$ are contained in $\operatorname{ran} c_k = \operatorname{ran} c - k$.

Let $\lambda \notin \operatorname{ran} c_k$. The solutions of the Dirichlet problem with non-homogeneous boundary condition are the solutions of

$$\begin{aligned} (A_0 - k - \lambda)u + av &= 0, , \\ bu + (c - k - \lambda)v &= 0, \\ \gamma_0 u &= \varphi. \end{aligned} \tag{4.10}$$

The second line is solved by $v = b(\lambda + k - c)^{-1}u$, which by insertion in the first line gives

$$\begin{aligned} (A_0 - k - \lambda + ab(\lambda + k - c)^{-1})u &= 0 \\ \gamma_0 u &= \varphi. \end{aligned} \tag{4.11}$$

When $\lambda \in \varrho(A_\gamma - k)$, the problem (4.11) has a unique solution $u = K^{k,\lambda}_\gamma \varphi \in \ker(A_0 - k - \lambda + ab(\lambda + k - c)^{-1})$ for each $\varphi \in H^{-\frac12}$, and (4.10) has the solution $\{u, b(k + \lambda - c)^{-1}u\} \in \ker(A_\gamma - k - \lambda)$. The Dirichlet-to-Neumann operator family for $A - k$ is

$$P^{k,\lambda} = \nu_1 \operatorname{pr}_1 K^{k,\lambda}_\gamma, \tag{4.12}$$

which identifies with the Dirichlet-to-Neumann operator family $P^{k,\lambda}_{\gamma_0,\nu_1}$ for $A_0 - k$.

As above, let $\widetilde{A}$ be the realisation of A under the boundary condition (4.9). It corresponds as in Section 3.2 to

$$L^\lambda = C - P^{k,\lambda} = L + P^{k,0} - P^{k,\lambda} : H^{-\frac{1}{2}} \to H^{\frac{1}{2}}, \text{ where}$$
$$L = C - P^{k,0}, \quad D(L) = H^{\frac{3}{2}}.$$

Then there is an M-function going in the opposite direction and satisfying

$$M_L(\lambda) = -(L + P^{k,0} - P^{k,\lambda})^{-1} \tag{4.13}$$

when $\lambda \in \varrho(\widetilde{A} - k) \cap \varrho(A_\gamma - k)$. With a notation similar to (3.14), $M_L(\lambda)$ acts as follows:

$$M_L(\lambda) = P^{k,\lambda}_{\nu_1 - C\gamma_0, \gamma_0}. \tag{4.14}$$

Proposition 4.2 *There exists a point belonging to the essential spectrum of $\widetilde{A} - k$ (and of $A_\gamma - k$) at which $M_L(\lambda)$ is holomorphic.*

P r o o f. Let $\lambda_0 = c(x_0) - k$, for some x_0 where $c_0(x) \in K \setminus \{0\}$. Then λ_0 belongs to $\sigma_{\text{ess}}(A_\gamma - k)$ and to $\sigma_{\text{ess}}(\widetilde{A} - k)$. We shall show that $M_L(\lambda)$ can be extended holomorphically across λ_0 or a neighbouring point.

In view of (4.4),

$$a(x)b(x)(\lambda + k - c(x))^{-1} = a(x)b(x)(\lambda + k)^{-1}, \text{ for all } x \in \overline{\Omega}, \text{ all } \lambda \neq -k. \tag{4.15}$$

This implies that the problem (4.11) takes the form

$$\begin{aligned} (A_0 - k - \lambda + ab(\lambda + k)^{-1})u &= 0 \\ \gamma_0 u &= \varphi, \end{aligned} \tag{4.16}$$

for $\lambda \neq -k$, and this is obtained by reduction from the problem

$$\begin{aligned} (A_0 - k - \lambda)u + av &= 0, \\ bu - (\lambda + k)v &= 0, \\ \gamma_0 u &= \varphi, \end{aligned} \tag{4.17}$$

a Dirichlet problem for

$$A_1 = \begin{pmatrix} A_0 - k & a(x) \\ b(x) & -k \end{pmatrix}. \tag{4.18}$$

Similarly, the problem

$$(A - k - \lambda)w = 0, \quad \nu_1 u - C\gamma_0 u = \psi, \tag{4.19}$$

is equivalent with

$$(A_1 - \lambda)w = 0, \quad \nu_1 u - C\gamma_0 u = \psi. \tag{4.20}$$

So, $M_L(\lambda)$ defined above coincides with the analogous operator for A_1:

$$M_L(\lambda) = P^{\lambda}_{\nu_1 - C\gamma_0, \gamma_0}(A_1). \tag{4.21}$$

It is holomorphic on $\varrho(\widetilde{A}_1)$, where $\widetilde{A}_1$ is the realisation of A_1 defined by the boundary condition (4.9).

This detour via A_1 gives information about the possible holomorphic extensions of the $M_L(\lambda)$-function for $\widetilde{A} - k$. We infer from the general result of [13] that the Dirichlet realisation $A_{1,\gamma}$ of A_1, as well as the realisation $\widetilde{A}_1$, have essential spectra equal to $\{-k\}$. Moreover, their spectra are contained in a half-space, and are discrete outside the point $\{-k\}$.

So $\lambda_0 = c(x_0) - k$ is either in $\varrho(\widetilde{A}_1)$ or is one of the discrete eigenvalues of $\widetilde{A}_1$, and in any case there is a disk $B(\lambda_0, \delta)$ around it such that $M_L(\lambda)$ is holomorphic on $B(\lambda_0, \delta) \setminus \{\lambda_0\}$. Since $c(x_0) - k$ is not the only point in the connected set $K - k$, there will be a point x_0' such that $c(x_0') - k \in B(\lambda_0, \delta) \setminus \{\lambda_0\}$.

We can conclude that $M_L(\lambda)$ is holomorphic at $\lambda_0' = c(x_0') - k$, but the point belongs to the essential spectrum of $\widetilde{A} - k$ (and of $A_\gamma - k$). □

Remark 4.3 The hypothesis (4.4) can be replaced by a weaker hypothesis as follows: Assume that K is a compact connected subset of $\operatorname{ran} c$ containing more than one point. Let $\omega, \omega', \omega''$ be subsets of $\overline{\Omega}$ with $\operatorname{dist}(\omega, \overline{\Omega} \setminus \omega') > 0$, $\operatorname{dist}(\omega', \overline{\Omega} \setminus \omega'') > 0$, such that $K \subset c(\omega)$ and $\operatorname{dist}(K, c(\overline{\Omega} \setminus \omega')) > 0$. *Assume that ab vanishes on ω''.* Let $\eta \in C^\infty$ with $\eta = 1$ on $\overline{\Omega} \setminus \omega''$ and $\eta = 0$ on ω', and set $c' = \eta c$. Then, for all $\lambda \notin \operatorname{ran} c' - k$, $ab(\lambda + k - c)^{-1} = ab(\lambda + k - c')^{-1}$. The problems (4.10) and (4.19) can now be replaced by problems where c is replaced by c' , whose range is disjoint from K, so that there will be points in $K - k$ where the M-function is holomorphic.

4.2 ODE counterexample

Consider the 2×2 matrix-formed operator

$$A \begin{pmatrix} u \\ v \end{pmatrix} (x) = \begin{pmatrix} -u''(x) & a(x)v'(x) \\ b(x)u'(x) & c(x)v(x) \end{pmatrix}, \tag{4.22}$$

acting on pairs of functions u, v on the interval $[0, 1]$ and $a, b, c \in C^\infty([0, 1])$. Its formal adjoint is given by

$$A' \begin{pmatrix} u \\ v \end{pmatrix} (x) = \begin{pmatrix} -u''(x) & (\overline{b(x)}v(x))' \\ (\overline{a(x)}u(x))' & \overline{c(x)}v(x) \end{pmatrix}, \tag{4.23}$$

and there is the Green's formula

$$\left(A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) - \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, A' \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) = (\Gamma_1 u, \Gamma_0' v) - (\Gamma_0 u, \Gamma_1' v)$$

where

$$\Gamma_1 u = \begin{pmatrix} -u_1'(1) + a(1)u_2(1) \\ u_1'(0) - a(0)u_2(0) \end{pmatrix}, \ \Gamma_0 u = \Gamma_0' u = \begin{pmatrix} u_1(1) \\ u_1(0) \end{pmatrix}, \ \Gamma_1' v = \begin{pmatrix} -v_1'(1) - b(1)v_2(1) \\ v_1'(0) + b(0)v_2(0) \end{pmatrix} \tag{4.24}$$

and $u, v \in D(A_{\max}) = H^2(0, 1) \times H^1(0, 1)$.

Proposition 4.4 *We introduce the quantities*

$$Q(x, \lambda) = \frac{a(x) \left(\frac{b(x)}{\lambda - c(x)} \right)'}{-1 + \frac{a(x)b(x)}{\lambda - c(x)}}, \ \alpha(x) = \exp \left(\int_0^x Q(s, \lambda)\, ds \right), \ \beta(x) = \frac{\alpha(x)}{-1 + \frac{a(x)b(x)}{\lambda - c(x)}}, \tag{4.25}$$

and let y_1 and y_2 be two linearly independent solutions of

$$\varphi'' + \frac{1}{\alpha} \left[\frac{\alpha'^2}{4\alpha} - \frac{\alpha''}{2} - \beta\lambda \right] \varphi = 0 \tag{4.26}$$

satisfying the initial conditions $y_1(0) = 1, y_1'(0) = 0, y_2(0) = 0, y_2'(0) = 1$. Note that y_1 and y_2 depend on λ, but we suppress this in the notation.

(i) *We have*

$$\ker(A_{\max} - \lambda) = \left\{ \begin{pmatrix} c_1 y_1 + c_2 y_2 \\ \frac{b}{\lambda - c}(c_1 y_1' + c_2 y_2') \end{pmatrix} \,\middle|\, c_1, c_2 \in \mathbb{C} \right\}.$$

(ii) *Consider the operator A_1, the restriction of $A_{\max}$ to $\ker \Gamma_1$, i.e. subject to Neumann boundary conditions. The matrix of the M-function has the entries*

$$m_{11} = \frac{y_1(1)}{\left[\frac{a(1)b(1)}{\lambda - c(1)} - 1 \right] y_1'(1)}, \ m_{12} = \frac{1}{\left[1 - \frac{a(0)b(0)}{\lambda - c(0)} \right] y_1'(1)},$$
$$m_{21} = \frac{1}{\left[\frac{a(1)b(1)}{\lambda - c(1)} - 1 \right] y_1'(1)}, \ m_{22} = \frac{y_2'(1)}{\left[1 - \frac{a(0)b(0)}{\lambda - c(0)} \right] y_1'(1)}. \tag{4.27}$$

(iii) *Assume in addition that the function b vanishes identically on an open interval $I \subseteq (0, 1)$. Then there is a second operator $\widetilde{A}$ with different essential spectrum to A_1, but giving rise to the same M-function.*

Remark 4.5 Note that if $y_1'(1) = 0$, then y_1 is a Neumann eigenfunction for (4.26), and λ is an eigenvalue of the operator A_1. Apart from this case, singularities of the M-function only occur when the coefficient in (4.26) blows up, as terms of the form $\frac{a(x)b(x)}{\lambda - c(x)} - 1$ also appear there.

P r o o f. (i) $(A - \lambda)u = 0$ can be written as

$$-u_1'' - \lambda u_1 + a u_2' = 0, \; b u_1' + (c - \lambda) u_2 = 0.$$

Solving the second equation for u_2 and substituting this into the first gives

$$-u_1'' + a \left(\frac{b u_1'}{\lambda - c} \right)' - \lambda u_1 = 0. \tag{4.28}$$

Introducing α and β as in (4.25), the equation (4.28) simply becomes $(\alpha u_1')' - \beta \lambda u_1 = 0$. Moreover, introducing

$$\varphi(x) = \exp \int^x \frac{\alpha'}{2\alpha} u_1(x),$$

the equation can be written as

$$\varphi'' + \frac{1}{\alpha} \left[\frac{\alpha'^2}{4\alpha} - \frac{\alpha''}{2} - \beta \lambda \right] \varphi = 0 \tag{4.29}$$

and the kernel of $A_{\max} - \lambda$ has the form

$$\ker(A_{\max} - \lambda) = \left\{ \begin{pmatrix} c_1 y_1 + c_2 y_2 \\ \frac{b}{\lambda - c}(c_1 y_1' + c_2 y_2') \end{pmatrix} \;\middle|\; c_1, c_2 \in \mathbb{C} \right\}.$$

(ii) We now calculate the M-function for the operator subject to Neumann boundary conditions. For any $(u_1, u_2) \in \ker(A_{\max} - \lambda)$ we have

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} -u_1'(1) + a(1) u_2(1) \\ u_1'(0) - a(0) u_2(0) \end{pmatrix} = \begin{pmatrix} u_1(1) \\ u_1(0) \end{pmatrix}.$$

A simple calculation then gives (4.27).

(iii) We first note that the determinant of the principal symbol of $A - \lambda$ is given by $\xi^2(ab + c - \lambda)$ which is zero if $\lambda \in \operatorname{ran}(ab + c)$. Hence ellipticity of the system fails for $\lambda \in \operatorname{ran}(ab + c)$ and by [13], the essential spectrum of the operator equals $\operatorname{ran}(ab + c)$. Under the additional assumption that the function b vanishes identically on an open interval $I \subseteq (0, 1)$, we have that $\operatorname{ran}(c|_I)$ is contained in the essential spectrum of the operator. Let $\widetilde{c}$ be a C^∞-function on $[0, 1]$ that coincides with c on $[0, 1] \setminus I$. This gives rise to another operator $\widetilde{A}$ with

$$\widetilde{A} \begin{pmatrix} u \\ v \end{pmatrix} (x) = \begin{pmatrix} -u''(x) & a(x) v'(x) \\ b(x) u'(x) & \widetilde{c}(x) v(x) \end{pmatrix}. \tag{4.30}$$

Let $\widetilde{A}_1$ be the realisation of $\widetilde{A}$ subject to Neumann boundary conditions. Then $\operatorname{ran}(\widetilde{c}|_I)$ will lie in $\sigma_{\mathrm{ess}}(\widetilde{A}_1)$. Therefore, in general the essential spectrum of the operators A_1 and $\widetilde{A}_1$ will differ. However, for the calculation of the M-function, c only appears in terms of the form $\frac{b(x)}{\lambda - c(x)}$ and by our assumptions,

$$\frac{b(x)}{\lambda - c(x)} = \frac{b(x)}{\lambda - \widetilde{c}(x)},$$

so the M-functions for A_1 and $\widetilde{A}_1$ coincide. Thus, we have another example where two operators with differing essential spectra give rise to the same M-function. □

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